Efficient Estimation in Two-Sided Truncated Location Models

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Abstract For a family of two-sided truncated location distributions, based on the generalized Neyman-Pearson lemma, an upper bound for the asymptotic distributions of the absolute deviations of all asymptotically median unbiased estimators for the location parameter is established, upon which the asymptotic efficiency is defined. Except for the cases in which the density has the same values at the truncation points, it is shown that there is no asymptotically median unbiased estimator to be two-sided asymptotically efficient. An adaptive asymptotically weak admissible median unbiased estimator of the location parameter is also constructed.

1 Introduction

The adaptiveness and the asymptotic efficiency are very important concepts in the theory of statistical estimation. Extensive research has been done when the underlying distributions have common support. See Ibragimov and Hasminski(1981), Akahira and Takeuchi (1981), and Bickel et al.(1998) for detailed discussion on these topics. Starting from 1970s, the research on the adaptiveness and the asymptotic efficiency in non-regular models, in particular, when the underlying distributions are not commonly supported, began to emerge. Early works on this topic were summarized in Akahira and Takeuchi (2003) and the references therein. For the unknown parameter θ in a class of uniformly distributed distribution family, Akahira (1982) successfully constructed an upper bound of the asymptotic distributions of $n(\hat{\theta}_n - \theta)$ for all asymptotically median unbiased (AMU, which will be defined later) estimators $\hat{\theta}_n$ of θ using the Neyman-Pearson testing framework. The concept of two-sided asymptotic efficiency is thus defined based on this bound and some examples were supplied. Akahira (1982) also noticed that for some examples, the proposed AMU estimators only attain the bound at one point, or are uniformly"close" to the bound. It is not clear, however, whether there exist any AMU estimators to be two-sided asymptotically efficient.

To be specific, let X be a random variable with distribution $P_\theta, \theta \in \Theta$. The parameter space Θ is assumed to be an open set in $\mathbb{R}$. Denote $\hat{\theta}_n$ an estimator of θ based on a sample $X_1, X_2, \ldots, X_n$ of size n from X. Let $\{c_n\}$ be a sequence of positive numbers tending to infinity as $n \to \infty$. Then $\hat{\theta}_n$ is called a consistent estimator of order $\{c_n\}$ if for every $\varepsilon > 0$ and every $\vartheta \in \Theta$ there exists a sufficiently small

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number $\delta > 0$ and a sufficiently large number L satisfying the following inequality

$$\limsup_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} P_\theta\{c_n|\hat{\theta}_n - \theta| \geq L\} < \varepsilon. \tag{1}$$

A cumulative distribution function $F_\theta(\cdot)$ is called the asymptotic distribution function of $c_n(\hat{\theta}_n - \theta)$, if for each real number t, $F_\theta(t)$ is continuous in θ, and for any $\vartheta \in \Theta$ there exists a positive number δ such that for any continuity point t of $F_\theta(\cdot)$,

$$\limsup_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \left|P_\theta\{c_n(\hat{\theta}_n - \theta) \leq t\} - F_\theta(t)\right| < \varepsilon.$$

Note that some requirements, such as the uniform requirement of $\sup_{\theta:|\theta-\vartheta|<\delta}$, do not present in the usual definitions of consistency and asymptotic distribution with order $\{c_n\}$. An estimator $\hat{\theta}_n$ is called to be an asymptotically median unbiased (A-MU) if for every $\vartheta \in \Theta$, there exists a positive number δ such that

$$\limsup_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \left|P_\theta\{\hat{\theta}_n \leq \theta\} - \frac{1}{2}\right| = 0,$$

$$\limsup_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} \left|P_\theta\{\hat{\theta}_n \geq \theta\} - \frac{1}{2}\right| = 0.$$

For the class of AMU estimators of θ, Akahira (1982) proposed the following left-hand side and right-hand side asymptotic efficiency

Definition 1. An AMU estimator $\hat{\theta}_n$ is called right-hand side (left-hand side) asymptotically efficient if for any AMU estimator $\tilde{\theta}_n$,

$$\liminf_{n\to\infty} \left[P_\theta\{c_n(\hat{\theta}_n - \theta) \leq t\} - P_\theta\{c_n(\tilde{\theta}_n - \theta) \leq t\}\right] \geq 0, \quad \text{for all } t > 0$$

$$(\liminf_{n\to\infty} \left[P_\theta\{c_n(\tilde{\theta}_n - \theta) \leq t\} - P_\theta\{c_n(\hat{\theta}_n - \theta) \leq t\}\right] \geq 0, \quad \text{for all } t < 0).$$

The above definition is intuitively well defined, but in practice, to use the definition as a criterion to check an AMU estimator to be left-hand or right-hand side asymptotically efficient, we have to find a tangible upper bound for $P_\theta\{c_n(\tilde{\theta}_n - \theta) \leq t\}$ when $t > 0$, $P_\theta\{c_n(\tilde{\theta}_n - \theta) \geq t\}$ when $t < 0$, for all AMU estimators $\tilde{\theta}_n$ of θ. For some particular distribution families, such an upper bound is constructed based on the Neyman-Pearson lemma after properly setting up a simple versus simple hypothesis testing problem about the unknown parameter. The detailed derivation of the upper bound can be found in Akahira (1982). In some non-regular cases, Akahira (1982) also showed that there exist either right-hand side asymptotically efficient estimator or left-hand side asymptotically efficient estimators. However, in general there are no AMU estimators to be both right-hand and left-hand side asymptotically efficient. See Takeuchi (1974) for some examples.

A weaker version than both right-hand and left-hand side asymptotic efficiency is the following two-sided asymptotic efficiency.

Definition 2. An AMU estimator $\hat{\theta}_n$ of θ is called two-sided asymptotically efficient if for any AMU estimator $\tilde{\theta}_n$ and $t > 0$,

$$\liminf_{n\to\infty}\left[P_\theta\{c_n|\hat{\theta}_n-\theta|\le t\}-P_\theta\{c_n|\tilde{\theta}_n-\theta|\le t\}\right]\ge 0.$$

For a special case in which X, or some transformation of X, is uniformly distributed over $[a(\theta),b(\theta)]$, $a(\theta)<b(\theta)$ and $a'(\theta)\le b'(\theta)<0$, where $a'(\theta)$ and $b'(\theta)$ are the derivatives of $a(\theta)$ and $b(\theta)$ with respect to θ, respectively, Akahira (1982) proposed an upper bound for $\limsup_{n\to\infty}P_\theta\{n|\hat{\theta}_n-\theta|\le t\}$, based on the generalized Neyman-Pearson lemma. For a uniform distribution and a symmetric truncated normal distribution, Akahira (1982) showed that $[X_{(1)}+X_{(n)}]/2$ is two-sided asymptotically efficient, while in a truncated exponential distribution case and an asymmetric truncated normal distribution case, two estimators are considered, but they are not asymptotically efficient, although for some t values, $\limsup_{n\to\infty}P_\theta\{n|\hat{\theta}_n-\theta|\le t\}$ attains the upper bound. The existence of asymptotically efficient estimator in such distribution families has not been answered.

In this paper, we shall focus on a class of truncated location family which subsume the truncation family discussed in Akahira (1982). By adopting the Neyman-Pearson testing framework, the left-hand side, right-hand side, and two-sided efficiency are discussed. The question of the existence of two-sided efficient estimation will be completely addressed in the paper, based on the newly defined notion of the asymptotically weak inadmissible median unbiased estimator. In fact, it is shown that in this particular distribution family, whether or not there exist asymptotically efficient estimators is totally determined by whether or not the density function at both ends are equal.

The paper is organized as follows. The model of interest and technical assumptions are laid out in Section 2, together with an important lemma and two examples. Section 3 includes the main results on one-sided asymptotic efficiency and the two-sided efficiency is discussed in Section 4; all the technical proofs are deferred to Section 5.

2 Models and Assumptions

Let $\mathbf{X}=\{X_1,X_2,\ldots,X_n\}$ be a sample of size n from a truncated location distribution

$$dP_\theta(x)=f(x-\theta)I_{[\theta+a,\theta+b]}(x)dx, \tag{2}$$

where a and b are two known real numbers, and θ is the unknown location parameter to be estimated. Here we do not have to assume f is known, even at the truncation points. Hence the model discussed here is essentially a semiparametric model and is more flexible than the one discussed in Akahira (1982).

The following regularity conditions on model (2) are needed to set up the bounds for defining the one-sided and two-sided efficiencies.

Assumptions:

(A1). $f(x)$ is twice continuously differentiable on $[a,b]$, and $f(x) > 0$ on $[a,b]$.
(A2). $f(a)f(b) > 0$.
(A3). $0 \le I = \int_a^b [f'(x)]^2/f(x)dx < \infty$.

(A1) guarantees a Taylor expansion of $f(x)$ up to second order can be implemented and the second order term can be neglected in some integrals when n is large; Similar to Akahira (1982), we only consider all consistent estimators of θ of order $c_n = n$, see (1) for the definition of consistency. In fact, one can show that for the truncated distribution family (2), n is the largest consistency order under assumption (A2). In condition (A3), if the integration is positive, then the central limit theorem can be applied to derive the asymptotic power functions of the tests for the hypothesis proposed in the following discussion; if the integration is 0, which implies that $f(x)$ is constant almost everywhere with respect to the Lebesgue measure on $[a,b]$, that is, the underlying distribution of X is uniform on $[a,b]$. In this case, a randomized most powerful test is needed to calculate the power function. This is also the case discussed by Weiss and Wolfowitz (1968), Akahira (1982).

The following lemma will be frequently used in the proof of the main results stated in the next two sections.

Lemma 1. *Suppose the two-sided truncation model (2) satisfies condition (A1), (A2) and (A3) with strict inequality. Then for any real number $t > 0$,*

$$\prod_{i=1}^{n} \frac{f(X_i - \theta - t/n)}{f(X_i - \theta)} \to \exp(t[f(a) - f(b)]) \tag{3}$$

in probability conditioning on $A_n = \cap_{i=1}^n A_{ni}$, where $A_{ni} = \{a + \theta + t/n < X_i < b + \theta\}$ no matter the true parameter is θ or $\theta + t/n$, and

$$\prod_{i=1}^{n} \frac{f(X_i - \theta + t/n)}{f(X_i - \theta)} \to \exp(-t[f(a) - f(b)]) \tag{4}$$

in probability conditioning on $B_n = \cap_{i=1}^n B_{ni}$, where $B_{ni} = \{a + \theta < X_i < b + \theta - t/n\}$ no matter the true parameter is θ or $\theta - t/n$. Furthermore, for $t > 0$,

$$\sqrt{n}\left(\sum_{i=1}^{n} \log\left[\frac{f(X_i - \theta - t/n)}{f(X_i - \theta)} I_{A_{ni}}\right] - t[f(a) - f(b)]\right) \Longrightarrow N(0, \sigma^2(t)), \tag{5}$$

$$\sqrt{n}\left(\sum_{i=1}^{n} \log\left[\frac{f(X_i - \theta + t/n)}{f(X_i - \theta)} I_{B_{ni}}\right] + t[f(a) - f(b)]\right) \Longrightarrow N(0, \sigma^2(t)) \tag{6}$$

conditioning on A_n and B_n, respectively, where $\sigma^2(t) = t^2[I - (f(b) - f(a))^2]$.

Two examples are given below to illustrate the validity of (3) and (4) in Lemma 1.

Example 1. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. from a truncated exponential distribution $f(x-\theta) = ce^{-(x-\theta)} I(a \le x - \theta \le b)$, where $a < b$, and $c = (e^{-a} - e^{-b})^{-1}$. Note that $f(a) = ce^{-a}$, $f(b) = ce^{-b}$ and $f(a) - f(b) = 1$. For $t > 0$, and conditioning on $\{a + \theta + t/n < X_{(1)} \le X_{(n)} < b + \theta\}$, we simply have

$$\prod_{i=1}^{n} \frac{f(X_i - \theta - t/n)}{f(X_i - \theta)} = e^t = e^{t[f(a)-f(b)]},$$

no matter the true parameter is θ or $\theta + t/n$, and conditioning on $\{a + \theta < X_{(1)} \le X_{(n)} < b + \theta - t/n\}$, we simply have

$$\prod_{i=1}^{n} \frac{f(X_i - \theta + t/n)}{f(X_i - \theta)} = e^{-t} = e^{-t[f(a)-f(b)]},$$

no matter the true parameter is θ or $\theta - t/n$,

Example 2. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. from a truncated normal distribution distribution $f(x-\theta) = ce^{-(x-\theta)^2/2} I(a < x - \theta < b)$, where $a < b$ and $c = (\Phi(b) - \Phi(a))^{-1}$, Φ is the CDF of the standard normal variable. Note that $f(a) = ce^{-a^2/2}$, $f(b) = ce^{-b^2/2}$. For $t > 0$, and conditioning on $\{a + \theta + t/n < X_{(1)} \le X_{(n)} < b + \theta\}$, we can show that, in probability,

$$\prod_{i=1}^{n} \frac{f(X_i - \theta - t/n)}{f(X_i - \theta)} = \exp\left(-\frac{t^2}{2n} + \frac{t}{n}\sum_{i=1}^{n}(X_i - \theta)\right) \to e^{t[f(a)-f(b)]},$$

since, by law of large numbers, no matter the true parameter is θ or $\theta + t/n$, $n^{-1}\sum_{i=1}^{n}(X_i - \theta) \to f(a) - f(b)$ in probability as $n \to \infty$. Similarly one can obtain (4).

3 Left-hand and Right-hand Side Asymptotic Efficiency

Let $\hat{\theta}_n$ be a AMU estimator of θ, and define

$$B(t) = \begin{cases} \liminf_{n\to\infty} P_\theta\{n(\hat{\theta}_n - \theta) \ge t\}, \; t < 0, \\ \liminf_{n\to\infty} P_\theta\{n(\hat{\theta}_n - \theta) \le t\}, \; t \ge 0. \end{cases}$$

It is easy to see that the larger the value of $B(t)$, the smaller the deviation of the AMU estimator $\hat{\theta}_n$ from θ. To find an upper bound for $B(t)$. we consider the most powerful test for the following hypothesis,

$$H_0 : \theta = \theta_0 + \frac{t}{n}, \quad \text{v.s.} \quad H_1 : \theta = \theta_0.$$

Let $\phi_n(X)$ be the most powerful test determined by the Neyman-Pearson lemma with asymptotic significance level $1/2$, and denote $\beta(t)$ the asymptotic power function of ϕ_n. Then the optimality of most powerful test implies that $B(t) \le \beta(t)$ for any

$\theta \in \Theta$ and $t \in \mathbb{R}$. Thus, an AMU estimator $\hat{\theta}_n$ is left-hand side or right-hand side asymptotically efficient if and only if $B(t) = \beta(t)$ for all t.

The following theorem provides an explicit form for $\beta(t)$.

Theorem 1. *Suppose the two-sided truncation model (2) satisfies condition (A1), (A2) and (A3). Then*

$$\beta(t) = \begin{cases} 1, & t < -\frac{\log 2}{f(a)}, \\ 1 - e^{f(b)t} + \frac{1}{2}e^{[f(b)-f(a)]t}, & -\frac{\log 2}{f(a)} \le t \le 0, \\ 1 - e^{-f(a)t} + \frac{1}{2}e^{[f(b)-f(a)]t}, & 0 \le t \le \frac{\log 2}{f(b)}, \\ 1, & t > \frac{\log 2}{f(b)}. \end{cases} \tag{7}$$

The condition in Theorem 1 is similar to the one used in Weiss and Wolfowitz (1968). They also provide a sufficient condition on f to ensure the validity of (3). It is not difficult to modify their sufficient condition to fit the current setup.

In fact, the above result is also true when $f(a) = 0$, $f(b) > 0$, or $f(a) > 0$, $f(b) = 0$, see the proof of Theorem 1. In these cases, one can easily construct left-hand side or right-hand side asymptotically efficient estimators. However, there is no AMU estimator to be both left-hand side and right-hand side asymptotically efficient. Also see Takeuchi (1974) for some interesting examples. In particular, for our current setup, we can show that

Corollary 1. *If $f(a) = f(b)$ in model (2), then $\beta(t) = 1.5 - e^{f(a)|t|}$ for $|t| \le log2/f(a)$, and 1 otherwise. Furthermore, we claim that $\hat{\theta}_{1/2} = (X_{(1)} + X_{(n)} - a - b)/2$ is not left-hand side or right-hand side asymptotically efficient.*

4 Two-Sided Asymptotic Efficiency

In this section we shall discuss the two-sided asymptotic efficiency of AMU estimators in model (2). Denote θ_0 the true value of θ, and for each $t > 0$, $\theta_1 = \theta_0 + t/n$, $\theta_2 = \theta_0 - t/n$. Then in the neighborhood of θ_0, by a similar argument as in Akahira (1982), we can show that an upper bound for $\limsup_{n\to\infty} P_\theta\{n|\hat{\theta}_n - \theta_0| < t\}$ is given by $\beta(t) = \limsup_{n\to\infty}(E_{\theta_2}\phi_n(X) - E_{\theta_1}\phi_n(X))$, and $\phi_n(X)$ is defined as

$$\phi_n(X) = \begin{cases} 1, & \prod_{i=1}^n f(X_i - \theta_2)I_{A_i} - \prod_{i=1}^n f(X_i - \theta_1)I_{B_i} > \lambda_n \prod_{i=1}^n f(X_i - \theta_0)I_{C_i}, \\ r, & \prod_{i=1}^n f(X_i - \theta_2)I_{A_i} - \prod_{i=1}^n f(X_i - \theta_1)I_{B_i} = \lambda_n \prod_{i=1}^n f(X_i - \theta_0)I_{C_i}, \\ 0, & \prod_{i=1}^n f(X_i - \theta_2)I_{A_i} - \prod_{i=1}^n f(X_i - \theta_1)I_{B_i} < \lambda_n \prod_{i=1}^n f(X_i - \theta_0)I_{C_i}, \end{cases} \tag{8}$$

where λ_n satisfies

$$\lim_{n\to\infty} E_{\theta_0}\phi_n(X) = \frac{1}{2} \tag{9}$$

and $A_i = \{X_i : a \le X_i - \theta_2 \le b\}$, $B_i = \{X_i : a \le X_i - \theta_1 \le b\}$, $C_i = \{X_i : a \le X_i - \theta_0 \le b\}$, $i = 1,2,\dots,n$. For the sake of completeness, we restate the definition of the two-sided asymptotically efficiency below.

Definition 3. An estimator $\hat{\theta}_n$ is called to be a two-sided asymptotic efficient estimator of θ in the distribution (2) if it satisfies

(1). $\hat{\theta}_n$ is an AMU estimator of θ with order n,
(2). $\limsup_{n\to\infty} P_\theta\{n|\hat{\theta}_n - \theta| < t\} = \beta(t)$ for all $t > 0$ and all $\theta \in \Theta$.

For the sake of brevity, denote $f_1 = \min\{f(a), f(b)\}$, $f_2 = \max\{f(a), f(b)\}$, $f^- = f(a) - f(b)$ and $f^+ = f(a) + f(b)$. The upper bound $\beta(t)$ is provided in the following theorem.

Theorem 2. *For the distribution family (1), when (A1), (A2) and (A3) hold, we have*

$$\beta(t) = \begin{cases} 1 - e^{-tf_2} + \frac{1}{2}e^{-tf^-}, & \text{if} \quad e^{-tf_1} - e^{-tf^+} \ge \frac{1}{2}, \\ 1 - e^{-2tf_2}, & \text{if} \quad e^{-tf_1} - e^{-tf^+} \le \frac{1}{2}, t \ge \frac{\log 2}{f_2}, \\ 1 - e^{-tf_2} + (1 - 2e^{-tf_2})\sinh(tf^-), & \text{if} \quad e^{-tf_1} - e^{-tf^+} \le \frac{1}{2}, \\ & \qquad e^{-tf_2} - e^{-tf^+} \le \frac{1}{2}, t < \frac{\log 2}{f_2}, \\ 1 - e^{-tf_1} + \frac{1}{2}e^{tf^-}, & \text{if} \quad e^{-tf_1} - e^{-tf^+} \le \frac{1}{2}, \\ & \qquad e^{-tf_2} - e^{-tf^+} > \frac{1}{2}, t < \frac{\log 2}{f_2}. \end{cases}$$

where $\sinh(x) = (e^x - e^{-x})/2$ *is the hyperbolic sine function.*

In particular, if $f(a) = f(b) > 0$, we have

Corollary 2. *In addition to the conditions in Theorem 2, we further assume that* $f(a) = f(b) > 0$ *in the distribution family (1). Then* $\beta(t) = 1 - e^{-2tf(a)}$.

Now, define

$$\hat{\theta}_n^* = \frac{1}{2}[X_{(1)} + X_{(n)} - a - b].$$

By the asymptotic independence of $X_{(1)}$ and $X_{(n)}$, one can show that, for any $t > 0$,

$$\lim_{n\to\infty} P_\theta\{n|\hat{\theta}_n^* - \theta_0| < t\} = 1 - e^{-2f(a)t} + \frac{f(a)}{f(a)+f(b)}[e^{-2f(a)t} - e^{-2f(b)t}].$$

In the case of $f(a) = f(b) > 0$, we can see that, for any $t > 0$,

$$\lim_{n\to\infty} P_\theta\{n|\hat{\theta}_n^* - \theta_0| < t\} = 1 - e^{-2f(a)t}$$

which equals to $\beta(t)$ given in Corollary 2. That is, $\hat{\theta}_n^*$ is two-sided asymptotically efficient when $f(a) = f(b)$. However, one can check that $\hat{\theta}_n^*$ is not two-sided asymptotically efficient in the case of $f(a) \neq f(b)$. In fact, we will show that no AMU estimator attains the upper bound $\beta(t)$, that is, no AMU estimator is two-sided asymptotically efficient in the sense of Definition 3. To prove our claim, the following definition is needed and the definition itself may not be just limited to the location model (2).

Definition 4. Let $X_1, X_2, \ldots, X_n$ be a sample from the distribution family $f(x,\theta)$, and denote $\mathscr{A}$ as the set of all the AMU estimators of θ. An estimator $\hat{\theta}_n \in \mathscr{A}$ is called to be an asymptotically weak inadmissible median unbiased estimator, if there exists an estimator $\tilde{\theta}_n \in \mathscr{A} - \{\hat{\theta}_n\}$ such that

$$\liminf_{n\to\infty} P_\theta\{c_n|\tilde{\theta}_n - \theta| < t\} \geq \limsup_{n\to\infty} P_\theta\{c_n|\hat{\theta}_n - \theta| < t\}$$

holds for all $\theta \in \Theta$, and $t > 0$; moreover, for every $\theta \in \Theta$, there exists a set A_θ with positive Lebesgue measure, the strict inequality holds for all $\theta \in \Theta$ and $t \in A_\theta$.

Without loss of generality, we shall assume that $f(a) > f(b)$. The main result we obtained is the following theorem.

Theorem 3. *Let $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ be a sample from the distribution family (2),then the solution $T_n(\mathbf{X})$ of the following equation*

$$\frac{\int_{X_{(n)}-b}^{T_n(\mathbf{X})} \prod_{i=1}^n f(X_i - \theta)d\theta}{\int_{X_{(n)}-b}^{X_{(1)}-a} \prod_{i=1}^n f(X_i - \theta)d\theta} = \frac{1}{2} \tag{10}$$

is an asymptotically weak admissible median unbiased estimator of θ, and $T_n(\mathbf{X})$ is equivalent to the solution $T_n^*(\mathbf{X})$ of the following equation

$$e^{nkT_n^*(\mathbf{X})} = \frac{1}{2}\left[e^{nk(X_{(n)}-b)} + e^{nk(X_{(1)}-a)}\right] \tag{11}$$

in the sense that $T_n(\mathbf{X})$ and $T_n^*(\mathbf{X})$ have the same asymptotic distribution, where $k = f(a) - f(b) > 0$. Moreover, we claim that there is no AMU estimator in $\mathscr{A}$ which is two sided asymptotically efficient.

As an example, we consider the following two-sided truncated exponential distribution family

$$f(x) = \begin{cases} ce^{-(x-\theta)}, & \theta \leq x \leq \theta + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $c = (1 - e^{-1})^{-1}$. Then the solution $T_n(\mathbf{X})$ defined in (10) has the form

$$T_n(\mathbf{X}) = \frac{1}{n}\log\frac{1}{2}\left[e^{nX_{(1)}} + e^{n(X_{(n)})-1}\right].$$

We can show that, as $n \to \infty$, $P_\theta\{|T_n(\mathbf{X}) - \theta| < t\}$ converges to

$$W(t) = \begin{cases} 1 - \frac{1}{2e^t}(2e^t - 1)^{-ce^{-1}} - \frac{1}{2e^{-t}}(2e^{-t} - 1)^c, & 0 \le t \le \log 2, \\ 1 - \frac{1}{2e^t}(2e^t - 1)^{-ce^{-1}}, & t > \log 2. \end{cases}$$

Akahira (1982) showed that among all AMU estimators having the form of $\hat{\theta}_p = pX_{(1)} + (1-p)(X_{(n)} - 1)$, the one with $p = e/(1+e)$ is the best in the sense that the larger the probability $P_\theta\{n|\hat{\theta}_p - \theta| < t\}$, the better the estimator, and for all $t \ge 0$,

$$L(t) = \lim_{n\to\infty} P_\theta\{n|\hat{\theta}_{e/(1+e)} - \theta| < t\} = 1 - e^{-c(1+e^{-1})t}.$$

The upper bound $\beta(t)$ in this two-sided truncated exponential distribution family is given by

$$\beta(t) = \begin{cases} 1 - e^{-2ct} - (1 - 2e^{-ct})\sinh(t), & 0 \le t \le \log 2/c, \\ 1 - e^{-2ct}, & t > \log 2/c. \end{cases}$$

For the easiness of comparison, the plots of the function $\beta(t)$, $W(t)$ and $L(t)$ for $t \in [0,3]$ are drawn in the left panel of Figure 1 below. As expected, $\beta(t)$, denoted by the solid line, is uniformly higher than the other two curves. The curves of $W(t)$, denoted by the dashed line, and $L(T)$, denoted by the dotted line, show that no one is uniformly higher than the other. This is clearer from the right panel of Figure 1 below, which magnifies the upper-right corner of the plot in the left panel. It is also evident that $W(t)$ is closer to the upper bound $\beta(t)$ than $L(t)$.

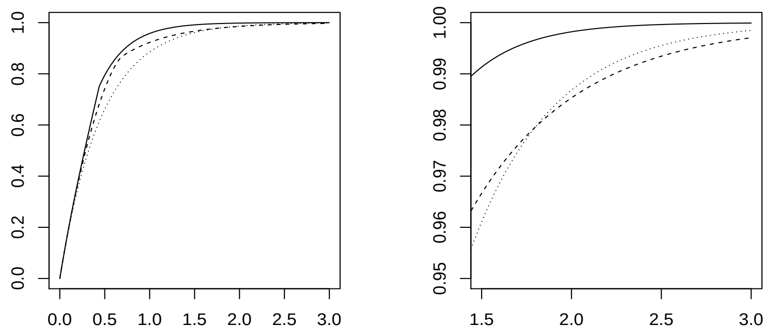

Figure 1: x-axis represents the t-values; y-axis represents the values of $\beta(t)$ (solid line), $W(t)$ (dashed line) and $L(t)$ (dotted line).

The estimator $T_n^*(\mathbf{X})$ defined in (11) only depends on the values of the density function f at both ends, and does not rely on the true form of f on (a,b), so it is an adaptive estimator. If both $f(a)$ and $f(b)$ are unknown, then we can estimate them by kernel technique. In fact, let K be a symmetric kernel density function, $K(a,b) = \int_0^{b-a} K(x)dx$, and define

$$\hat{f}_n(a) = \frac{1}{nhK(a,b)} \sum_{i=1}^{n} K\left(\frac{X_i - X_{(1)}}{h}\right), \quad \hat{f}_n(b) = \frac{1}{nhK(a,b)} \sum_{i=1}^{n} K\left(\frac{X_{(n)} - X_i}{h}\right).$$

Then one can show that $\hat{f}_n(a)$ and $\hat{f}_n(b)$ are consistent estimators of $f(a)$ and $f(b)$, respectively. Replace $f(a)$ and $f(b)$ in (11), one can get a two-sided adaptive admissible estimator of θ.

5 Proofs

Let's prove Lemma 1 first.

Proof of Lemma 1: Let us show (5) first. For convenience, let $Z_{ni} = \log \frac{f(X_i-\theta-t/n)}{f(X_i-\theta)} I_{A_{ni}}$. Then for each n, $\sum_{i=1}^n Z_{ni}$ is a sum of i.i.d. random variables no matter the true parameter is θ or $\theta + t/n$, conditioning on $a+\theta+t/n \le X_{(1)} \le X_{(n)} \le b+\theta$ and $a+\theta \le X_{(1)} \le X_{(n)} \le b+\theta - t/n$. Let's assume the true parameter is θ. Then conditioning $a+\theta+t/n \le X_{(1)} \le X_{(n)} \le b+\theta$, $X_i, i = 1,2,\ldots,n$ are i.i.d. and have density function $cf(x-\theta)I(a+\theta+t/n \le x \le b+\theta)$, where $c^{-1} = P(a+\theta+t/n \le X \le b+\theta) = 1+O(1/n)$. In the following discussion, all expectations are calculated under this conditional distribution. For $k = 1,2,3$, we have

$$\begin{aligned}
EZ_{ni}^k &= c\int_{a+\theta+\frac{t}{n}}^{b+\theta} \left[\log \frac{f(x-\theta-t/n)}{f(x-\theta)}\right]^k f(x-\theta)dx = c\int_{a+\frac{t}{n}}^{b} \left[\log \frac{f(x-t/n)}{f(x)}\right]^k f(x)dx \\
&= c\int_{a+\frac{t}{n}}^{b} \left[\log f(x) - \frac{t}{n}\frac{f'(x)}{f(x)} + O\left(\frac{1}{n^2}\right) - \log f(x)\right]^k f(x)dx \\
&= \left(1+O\left(\frac{1}{n}\right)\right)\cdot\left[\int_{a+\frac{t}{n}}^{b}\left[-\frac{t}{n}\frac{f'(x)}{f(x)}\right]^k f(x)dx + O\left(\frac{1}{n^{2k}}\right)\right] \\
&= \frac{(-t)^k}{n^k}\int_a^b \left[\frac{f'(x)}{f(x)}\right]^k f(x)dx + O\left(\frac{1}{n^{k+1}}\right).
\end{aligned}$$

In particular,

$$EZ_{n1} = \frac{t[f(a)-f(b)]}{n} + O\left(\frac{1}{n^2}\right), \quad EZ_{n1}^2 = \frac{t^2 I}{n^2} + O\left(\frac{1}{n^3}\right), \quad EZ_{ni}^3 = O\left(\frac{1}{n^3}\right).$$

Therefore,

$$\frac{\sum_{i=1}^n E[Z_{ni}-EZ_{ni}]^3}{\left(\sqrt{\mathrm{Var}(\sum_{i=1}^n Z_{ni})}\right)^3} = \frac{nE[Z_{n1}-EZ_{n1}]^3}{\left(\sqrt{n\mathrm{Var}(Z_{ni})}\right)^3} = \frac{n\cdot O(n^{-3})}{[n\cdot O(n^{-2})]^{3/2}} = o(1).$$

By Lyapunov central limit theorem, we have

$$\frac{\sum_{i=1}^n Z_{ni} - nEZ_{n1}}{\sqrt{\mathrm{Var}(\sum_{i=1}^n Z_{ni})}} \Longrightarrow N(0,1)$$

in distribution. Further note that

$$\frac{nEZ_{n1} - t[f(a)-f(b)]}{\sqrt{\mathrm{Var}(\sum_{i=1}^n Z_{ni})}} = \frac{O(n^{-1})}{O(n^{-1/2})} = o(1), \quad \frac{\mathrm{Var}(\sum_{i=1}^n Z_{ni})}{n^{-1}\sigma^2(t)} \to 1,$$

we obtain that

$$\sqrt{n}\left[\sum_{i=1}^n Z_{ni} - t[f(a)-f(b)]\right] \Longrightarrow N(0,\sigma^2(t)).$$

So, (5) is proved, and hence (3). If the true parameter is $\theta + t/n$, then still $c = 1 + O(1/n)$. So for $k = 1,2,3$,

$$\begin{aligned}
EZ_{ni}^k &= c\int_{a+\frac{t}{n}}^b \left[\log\frac{f(x-t/n)}{f(x)}\right]^k f(x-t/n)dx \\
&= c\int_{a+\frac{t}{n}}^b \left[\log f(x) - \frac{t}{n}\frac{f'(x)}{f(x)} + O\left(\frac{1}{n^2}\right) - \log f(x)\right]^k \left[f(x) + O\left(\frac{1}{n}\right)\right] dx \\
&= \int_{a+\frac{t}{n}}^b \left[-\frac{t}{n}\frac{f'(x)}{f(x)}\right]^k f(x)dx + O\left(\frac{1}{n^{k+1}}\right) \\
&= \frac{(-t)^k}{n^k}\int_a^b \left[\frac{f'(x)}{f(x)}\right]^k f(x)dx + O\left(\frac{1}{n^{k+1}}\right).
\end{aligned}$$

Therefore, the proofs of (5) and (3) are the same as the one when θ is the true parameter. Similarly, one can show (6) and (4). The details are omitted for the sake of brevity. □

Proof of Theorem 1: First assume that $t > 0$. For a sample $X_1, X_2, \dots, X_n$ from model (2), denote $C_n = \{X_i, i = 1,2,\dots,n : X_{(1)} < \theta_0 + t/n + a\}$ and A_n as in Lemma 1. By Neyman-Pearson lemma, the UMP test of the hypothesis $H_0 : \theta_n = \theta_0 + t/n$ versus $H_1 : \theta_0$ has the following form

$$\varphi_n(X) = \begin{cases} 1, & \text{if } X \in C_n \bigcup [A_n \bigcap \{\prod_{i=1}^n f(X_i - \theta_0) > k_n \prod_{i=1}^n f(X_i - \theta_n)\}] \\ r_n, & A_n \bigcap \{\prod_{i=1}^n f(X_i - \theta_0) = k_n \prod_{i=1}^n f(X_i - \theta_n)\} \\ 0, & \text{otherwise} \end{cases}$$

where k_n is chosen so that

$$\lim_{n\to\infty} E_{\theta_n}\varphi_n(X) = \frac{1}{2}. \tag{12}$$

Based on the asymptotic result (4) in Lemma 1, a proper k_n can be chosen so that $r_n = 0$, that is, a non-randomized test can be constructed. Also $\lim_{n\to\infty} E_{\theta_n}\varphi_n(X)$ equals

$$\lim_{n\to\infty}\left[P_{\theta_n}(C_n)+P_{\theta_n}(A_n)P_{\theta_n}\left(\prod_{i=1}^n f(X_i-\theta_0)>k_n\prod_{i=1}^n f(X_i-\theta_n)\Big|A_n\right)\right]=\frac{1}{2}.$$

Since $P_{\theta_n}(C_n)=0$, $\lim_{n\to\infty}P(A_n)=e^{-f(b)t}$, so, we can choose k_n so that

$$P_{\theta_n}\left(\prod_{i=1}^n f(X_i-\theta_0)>k_n\prod_{i=1}^n f(X_i-\theta_n)\Big|A_n\right)\to\frac{1}{2}e^{f(b)t}, \tag{13}$$

if $0\le t\le(\log 2)/f(b)$. Also note that

$$\lim_{n\to\infty}P_{\theta_0}(C_n)=1-e^{-f(a)t},\quad \lim_{n\to\infty}P_{\theta_0}(A_n)=e^{-f(a)t},$$

and by Lemma 1, (13) also holds when the true parameter is θ_0. Therefore,

$$\lim_{n\to\infty}E_{\theta_0}\varphi_n(X)=1-e^{-f(a)t}+\frac{1}{2}e^{[f(b)-f(a)]t}. \tag{14}$$

If $t>\log 2/f(b)$, a most powerful test can be defined as

$$\varphi_n(X)=\begin{cases}1, & \text{if } X\in B_n\bigcup\left\{\theta_0+t/n+a\le X_{(1)}\le X_{(n)}\le\theta_0+u/n+b\right\}\\ 0, & \text{otherwise,}\end{cases}$$

where $0<u<t$. Note that

$$\begin{aligned}E_{\theta_n}\varphi_n(X)&=P_{\theta_n}\left\{\theta_0+t/n+a\le X_{(1)}\le X_{(n)}\le\theta_0+u/n+b\right\}\\&=\left[\int_{\theta_0+t/n+a}^{\theta_0+u/n+b}f(x-\theta_0-t/n)dx\right]^n=\left[1-\int_{(u-t)/n+b}^{b}f(x)dx\right]^n.\end{aligned}$$

So $\lim_{n\to\infty}E_{\theta_n}\varphi_n(X)=e^{-f(b)(t-u)}$. Choose $u=t-\log 2/f(b)$, then $\varphi_n(X)$ has the desired asymptotic level $1/2$. Accordingly, with such a u, the asymptotic power of $\varphi_n(X)$ at θ_0 is given by

$$\lim_{n\to\infty}E_{\theta_0}\varphi_n(X)=1. \tag{15}$$

For $t<0$, the MPT for testing $H_0:\theta_n=\theta_0+t/n$ versus $H_1:\theta_0$ has the following form

$$\varphi_n(X)=\begin{cases}1, & \text{if } X\in D_n\bigcup[B_n\bigcap\{\prod_{i=1}^n f(X_i-\theta_0)>k_n\prod_{i=1}^n f(X_i-\theta_n)\}]\\ r_n, & B_n\bigcap\{\prod_{i=1}^n f(X_i-\theta_0)=k_n\prod_{i=1}^n f(X_i-\theta_n)\}\\ 0, & \text{otherwise,}\end{cases}$$

where $D_n=\{X_i:i=1,2,\ldots,n:X_{(n)}>\theta_0+t/n+b\}$, and $B_n=\{X_i:i=1,2,\ldots,n:\theta_0+a\le X_{(1)}\le X_{(n)}\le\theta_0+t/n+b\}$. A similar argument as before shows that, under the constraint (12), for $-\log 2/f(a)\le t\le 0$, the power function satisfies

$$\lim_{n\to\infty} E_{\theta_0}\varphi_n(X) = 1 - e^{f(b)t} + \frac{1}{2}e^{[f(b)-f(a)]t}. \tag{16}$$

If $t < -\log 2/f(a)$, we can choose $u = t + \log 2/f(a)$ and the following test

$$\varphi_n(X) = \begin{cases} 1, & \text{if } X \in D_n \bigcup \{\theta_0 + u/n + a \le X_{(1)} \le X_{(n)} \le \theta_0 + t/n + b\} \\ 0, & \text{otherwise,} \end{cases}$$

is most powerful with the asymptotic level $1/2$ and the asymptotic power 1. This, together with the results (14), (16) and (15), completes the proof of Theorem 1. □

Proof of Corollary 1: The form of $\beta(t)$ is an immediate consequence of Theorem 1 with $f(a) = f(b)$ in (7). When $f(a) = f(b)$, we can also show that

$$\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \le t\} = \begin{cases} \frac{1}{2}e^{2f(a)t}, & t < 0, \\ 1 - \frac{1}{2}e^{-2f(a)t}, & t \ge 0. \end{cases}$$

It is easy to see that $\hat{\theta}_{1/2}$ is an AMU estimator, and for $0 < t < \log 2/f(a)$,

$$\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \le t\} = 1 - \frac{1}{2}e^{-2f(a)t} < 1.5 - e^{-f(a)t},$$

and for $t \ge \log 2/f(a)$, $\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \le t\} = 1 - \frac{1}{2}e^{-2f(a)t} < 1$. For $-\log 2/f(a) < t < 0$,

$$\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \ge t\} = 1 - \frac{1}{2}e^{2f(a)t} < 1.5 - e^{f(a)t},$$

and for $t < -\log 2/f(a)$, $\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \ge t\} = 1 - \frac{1}{2}e^{2f(a)t} < 1$. That is, for all $t > 0$, $\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \le t\} < \beta(t)$, and for all $t < 0$, $\lim_{n\to\infty} P_\theta\{n(\hat{\theta}_{1/2} - \theta) \ge t\} < \beta(t)$. One can see that the equality holds only at $t = 0$. This concludes the proof of Corollary 1. □

Proof of Theorem 2: Without loss of generality, we only prove the result for $f(a) > f(b) > 0$.

The proof will be divided into three parts based on $\lambda_n < 0$, $\lambda_n = 0$ and $\lambda_n > 0$ in (8). Since the proofs are similar, only the proof for $\lambda_n > 0$ is present here for the sake of brevity. Denote

$$A_n = \{X_{(1)} < \theta_0 + a, X_{(n)} < \theta_2 + b\}, \quad B_n = \{\theta_0 + a < X_{(1)} < \theta_1 + a, X_{(n)} < \theta_0 + b\},$$

$$C_n = \{X_{(1)} > \theta_1 + a, \theta_2 + b < X_{(n)} < \theta_0 + b\}, \quad D_n = \{\theta_1 + a < X_{(1)} \le X_{(n)} < \theta_2 + b\},$$

and $E_n = \{\theta_1 + a < X_{(1)} \le X_{(n)} < \theta_0 + b\}$. Then, $\phi_n^* = 1$ if and only if $X_1, X_2, \ldots, X_n$ belongs to

$$A_n \bigcup B_n \bigcup \left[D_n \bigcap \left\{ \frac{\prod_{i=1}^n f(X_i - \theta_2)}{\prod_{i=1}^n f(X_i - \theta_0)} - \frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} > \lambda_n \right\} \right]$$
$$\bigcup \left[C_n \bigcap \left\{ \frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} < -\lambda_n \right\} \right].$$

Similarly to the proof of Lemma 1, we can show that under θ_0, θ_1 and θ_2, in probability,

$$\frac{\prod_{i=1}^n f(X_i - \theta_2)}{\prod_{i=1}^n f(X_i - \theta_0)} I[\theta_1 + a < X_i < \theta_2 + b] \to e^{-tf^-},$$
$$\frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} I[\theta_1 + a < X_i < \theta_2 + b] \to e^{tf^-}.$$

as $n \to \infty$. This is also true after replacing $I[\theta_1 + a < X_i < \theta_2 + b]$ with $I[\theta_1 + a < X_i < \theta_0 + b]$. After certain normalization, asymptotical normalities can also be achieved as in Lemma 1.

If $t < \log 2/f(a)$ and $e^{-f(a)t} - e^{-tf^+} > 1/2$, we can choose λ_n such that $\lim_{n\to\infty} \lambda_n = -e^{-tf^-}$ and

$$\lim_{n\to\infty} P_{\theta_0} \left\{ \frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} \le -\lambda_n \Big| C_n \right\} = \frac{1 - e^{f(a)t}/2 - e^{-f(b)t}}{1 - e^{-f(b)t}}.$$

Since $e^{tf^-} - e^{-tf^-} < e^{tf^-}$, with such a choice of λ_n, we will have

$$\lim_{n\to\infty} P_{\theta_0} \left\{ \frac{\prod_{i=1}^n f(X_i - \theta_2)}{\prod_{i=1}^n f(X_i - \theta_0)} - \frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} > \lambda_n \Big| D_n \right\} = 1.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} E_{\theta_0} \phi_n^*(X) &= 1 - e^{-f(a)t} \\ &\quad + e^{-tf^+} \cdot \lim_{n\to\infty} P_{\theta_0} \left\{ \frac{\prod_{i=1}^n f(X_i - \theta_2)}{\prod_{i=1}^n f(X_i - \theta_0)} - \frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} > \lambda_n \Big| D_n \right\} \\ &\quad + \left[e^{-f(a)t} - e^{-tf^+} \right] \cdot \lim_{n\to\infty} P_{\theta_0} \left\{ \frac{\prod_{i=1}^n f(X_i - \theta_1)}{\prod_{i=1}^n f(X_i - \theta_0)} \le -\lambda_n \Big| C_n \right\} \\ &= \frac{1}{2}.\end{aligned}$$

It is also easy to check that

$$\lim_{n\to\infty}(E_{\theta_2}\phi_n^*(X)-E_{\theta_1}\phi_n^*(X)) = 1-e^{-f(a)t}+e^{-f(a)t}-e^{-2f(a)t}$$
$$+(e^{-2(a)t}-e^{-2f(b)t})\cdot\lim_{n\to\infty}P_{\theta_1}\left\{\frac{\prod_{i=1}^n f(X_i-\theta_2)}{\prod_{i=1}^n f(X_i-\theta_0)}-\frac{\prod_{i=1}^n f(X_i-\theta_1)}{\prod_{i=1}^n f(X_i-\theta_0)}>\lambda_n\Big|D_n\right\}$$
$$-(e^{-f(b)t}-e^{-2f(b)t})\cdot\lim_{n\to\infty}P_{\theta_1}\left\{\frac{\prod_{i=1}^n f(X_i-\theta_1)}{\prod_{i=1}^n f(X_i-\theta_0)}\le-\lambda_n\Big|C_n\right\}$$
$$= \ 1-e^{-f(b)t}+\frac{1}{2}e^{tf^-}. \tag{17}$$

If $t<\log 2/f(a)$ and $e^{-f(a)t}-e^{-tf^+}\le 1/2$, we can choose λ_n such that $\lim_{n\to\infty}\lambda_n = e^{tf^-}-e^{-tf^-}$ and

$$\lim_{n\to\infty}P_{\theta_0}\left\{\frac{\prod_{i=1}^n f(X_i-\theta_2)}{\prod_{i=1}^n f(X_i-\theta_0)}-\frac{\prod_{i=1}^n f(X_i-\theta_1)}{\prod_{i=1}^n f(X_i-\theta_0)}>\lambda_n\Big|D_n\right\}=e^{f(b)t}-\frac{1}{2}e^{tf^+}.$$

Since $e^{tf^-}-e^{-tf^-}<e^{tf^-}$, with such a choice of λ_n, we will have

$$\lim_{n\to\infty}P_{\theta_0}\left\{\frac{\prod_{i=1}^n f(X_i-\theta_1)}{\prod_{i=1}^n f(X_i-\theta_0)}\le-\lambda_n\Big|C_n\right\}=0.$$

Therefore, we still have $\lim_{n\to\infty}E_{\theta_0}\phi_n^*(X)=1/2$, but

$$\lim_{n\to\infty}(E_{\theta_2}\phi_n^*(X)-E_{\theta_1}\phi_n^*(X))=1-e^{-2f(a)t}+(1-2e^{-f(a)t})\sinh[tf^-]. \tag{18}$$

The similar argument can be extended to the cases of $\lambda_n=0$ and $\lambda_n>0$ by looking for a proper critical function ϕ_n^*. For the sake of brevity, the proofs are left out. □.

Remark: If the asymptotic distribution of the likelihood ratios, after normalization, are degenerated, then for $t<\log 2/f(a)$ and $e^{-f(a)t}-e^{-tf^+}>1/2$, we can choose $\phi_n^*(X)=1-I[F_n\cup G_n]$, where $F_n=\{X_{(1)}>\theta_0+a+u/n,\theta_2+b<X_{(n)}<\theta_0+b\}$, $G_n=\{X_{(1)}>\theta_0+a,\theta_0+b<X_{(n)}<\theta_1+b\}$, and $0<u<t$ such that $E_{\theta_0}\phi_n^*\to 1/2$. Then we also have (17). If $t<\log 2/f(a)$ and $e^{-f(a)t}-e^{-tf^+}\le 1/2$, then we can choose $\phi_n(X)$ to be the indicator function of the union of three sets: $\{X_{(1)}<\theta_0+a,X_{(n)}<\theta_2+b\}$, $G_n=\{\theta_0+a<X_{(1)}<\theta_1+a,X_{(n)}<\theta_0+b\}$ and $\{\theta_1+a<X_{(1)}<\theta_0+u/n+a,X_{(n)}<\theta_2+b\}$, where u is chosen so that $u\ge t$ and $E_{\theta_0}\phi_n^*\to 1/2$. With such choices of ϕ_n^* and u, we can obtain (18).

Proof of Theorem 3: By Fox and Rubin (1964), the roof $T_n(\mathbf{X})$ of equation (10) is an AMU and an admissible estimator of θ under the absolute deviation loss $L(\theta,t)=|t-\theta|$. Suppose there is another AMU estimator $\hat{\theta}_n$ of θ such that

$$\liminf_{n\to\infty}P_\theta\{n|\hat{\theta}_n-\theta|<t\}\ge\limsup_{n\to\infty}P_\theta\{n|T(X_{(1)},X_{(n)})-\theta|<t\}$$

or equivalently,

$$\limsup_{n\to\infty} P_\theta\{n|\hat{\theta}_n - \theta| > t\} \le \liminf_{n\to\infty} P_\theta\{n|T(X_{(1)}, X_{(n)}) - \theta| > t\}$$

holds for all $\theta \in \Theta$ and $t > 0$. Moreover, for each $\theta \in \Theta$, there exists a set A_θ with positive Lebesgue measure such that the strict inequality holds for all $t \in A_\theta$. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} E_\theta[n|\hat{\theta}_n - \theta|] = \limsup_{n\to\infty} \int_0^\infty P_\theta\{n|\hat{\theta}_n - \theta| > t\}dt \\
\le &\int_0^\infty \limsup_{n\to\infty} P_\theta\{n|\hat{\theta}_n - \theta| > t\}dt < \int_0^\infty \liminf_{n\to\infty} P_\theta\{n|T_n(\mathbf{X}) - \theta| > t\}dt \\
\le &\liminf_{n\to\infty} \int_0^\infty P_\theta\{n|T_n(\mathbf{X}) - \theta| > t\}dt = \liminf_{n\to\infty} E_\theta[n|T_n(\mathbf{X}) - \theta|]
\end{aligned}$$

which implies, when n is large enough, $E_\theta|\hat{\theta}_n - \theta| < E_\theta|T_n(\mathbf{X}) - \theta|$. This contradicts the admissibility of $T_n(\mathbf{X})$, and therefore, $T_n(\mathbf{X})$ must be an asymptotically weak admissible median unbiased estimator.

To show the equivalence of $T_n(\mathbf{X})$ and $T_n^*(\mathbf{X})$ defined in (11), note that by Taylor expansion

$$\frac{\prod_{i=1}^n f(X_i - \theta)}{\prod_{i=1}^n f(X_i)} = \exp\left[\sum_{i=1}^n \log \frac{f(X_i - \theta)}{f(X_i)}\right] = \exp\left[\theta \sum_{i=1}^n \frac{f'(X_i)}{f(X_i)} + o_p(n\theta)\right],$$

then from $\theta \in [X_{(n)} - b, X_{(1)} - a]$ which implies that $\theta = O_p(1/n)$, then $o_p(n\theta) = o_p(1)$, therefore, we obtain

$$\frac{1}{2} = \frac{\int_{X_{(n)}-b}^{T_n(\mathbf{X})} \prod_{i=1}^n f(X_i - \theta)d\theta}{\int_{X_{(n)}-b}^{X_{(1)}-a} \prod_{i=1}^n f(X_i - \theta)d\theta} = \frac{\int_{X_{(n)}-b}^{T_n(\mathbf{X})} \exp[-\theta \sum_{i=1}^n f'(X_i)/f(X_i)]d\theta}{\int_{X_{(n)}-b}^{X_{(1)}-a} \exp[-\theta \sum_{i=1}^n f'(X_i)/f(X_i)]d\theta}[1 + o_p(1)].$$

Denote $M_n(X) = n^{-1}\sum_{i=1}^n f'(X_i)/f(X_i)$, then the above equality indeed is equivalent to

$$\frac{\exp[-n(X_{(n)} - b)M_n(X)] - \exp[-nT_n(\mathbf{X})M_n(X)]}{\exp[-n(X_{(n)} - b)M_n(X)] - \exp[-n(X_{(1)} - a)M_n(X)]}[1 + o_p(1)] = \frac{1}{2},$$

or

$$e^{-nT_n(\mathbf{X})M_n(X)} = \frac{1}{2}\left[e^{-n(X_{(n)}-b)M_n(X)} + e^{-n(X_{(1)}-a)M_n(X)}\right] + o_p(1).$$

The the desired result follows from the facts that $M_n(X) \to E_0[f'(X)/f(X)] = f(b) - f(a)$ in probability.

The proof of the theorem will be complete if we can show that $T_n(\mathbf{X})$ defined above is not two-sided asymptotic efficient. For this purpose, the asymptotic distribution of $T_n(\mathbf{X})$ should be derived. The equivalence of $T_n(\mathbf{X})$ and $T_n^*(\mathbf{X})$ implies that it suffices to discuss $T_n^*(\mathbf{X})$ only. Without loss of generality, let's assume that $\theta = 0$. It is easy to see that

$$P_0\left\{2^{-1}e^{nk[X_{(n)}-b]}<x\right\}=\begin{cases}0, & x<0,\\ \left[1+\frac{f(b)\log 2x}{nk}+o\left(\frac{1}{n}\right)\right]^n\to(2x)^{f(b)/k}, & 0\le x\le 1/2,\\ 1, & x>1/2.\end{cases}$$

and

$$P_0\left\{2^{-1}e^{nk[X_{(1)}-a]}<x\right\}=\begin{cases}1-\left[1-\frac{f(a)\log 2x}{nk}+o\left(\frac{1}{n}\right)\right]^n\to(2x)^{-f(a)/k}, & x\ge 1/2,\\ 0, & x<1/2.\end{cases}$$

Based on these two probabilities, and also the asymptotic independence of $X_{(1)}$ and $X_{(n)}$, we can obtain

$$\lim_{n\to\infty}P_\theta\{n|T_n(\mathbf{X})-\theta|<t\}=\lim_{n\to\infty}P_\theta\{nk|T_n^*(\mathbf{X})-\theta|<kt\}$$
$$=\begin{cases}1-\frac{1}{2e^{kt}}(2e^{kt}-1)^{-f(b)/f^-}-\frac{1}{2e^{-kt}}(2e^{-kt}-1)^{f(a)/f^-}, & 0\le t\le\frac{\log 2}{k},\\ 1-\frac{1}{2e^{kt}}(2e^{kt}-1)^{-f(b)/f^-}, & t>\frac{\log 2}{k}.\end{cases}$$

We can check that this limit indeed does not exceed the $\beta(t)$ function for all $t\ge 0$, and when t is big enough, it is also easy to see that this limit is strictly less than $\beta(t)$. That is, we proved that $T_n(\mathbf{X})$ is not two-sided asymptotically efficient. This, together with the fact that $T_n(\mathbf{X})$ is an asymptotically weak admissible median unbiased estimator, implies that there is no AMU in the location family (2) to be two-sided asymptotically efficient. □

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