— WORKSHOP —

Applied Classical and Modern Multivariate Statistical Analysis

Module 4: Inferences on Means and Covariances

Weixing Song, Juan Du

Department of Statistics
Kansas State University

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Outline

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1. Inference on the Normal Population Mean: One Population Case
   - Test
   - Confidence Regions and Simultaneous Comparisons
   - Large Sample Inferences

2. Inference on Normal Population Means: Several Populations Case
   - Paired Comparison
   - Repeated Measures Design for Comparing Treatments
   - Comparing Mean Vectors from Two Populations
   - Comparing Several Multivariate Population Means

3. Testing of Equality of Covariance Matrices

Hypothesis and Test Statistic

Question of interest: Suppose $X \sim N_p(\mu, \Sigma)$. We would like to test

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

Recall what we have done in the univariate case.

Hypothesis and Test Statistic

Question of interest: Suppose $X \sim N_p(\mu, \Sigma)$. We would like to test

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

Recall what we have done in the univariate case.

- Test statistic
$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}.$$
- Test criterion: Reject H_0 if
$$|t| \geq t_{n-1}(\alpha/2) \quad \text{or} \quad t^2 \geq t^2_{n-1}(\alpha/2) = F_{1,n-1}(\alpha).$$

In the multivariate case, the test statistic is the Hotelling's T^2:

$$T^2 = n(\bar{\mathbf{X}} - \mu_0)'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \mu_0).$$

Sampling Distribution of The Test Statistic

Sampling Distribution of T^2

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample from an $N_p(\mu, \Sigma)$ population. Then

$$n(\bar{\mathbf{X}} - \mu)'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \mu) \sim \frac{(n-1)p}{n-p} F_{p,n-p},$$

where $F_{p,n-p}$ is an F distribution with the first degrees of freedom p and the second degrees of freedom $n-p$.

NOTE: It would be beneficial to remember the following formula:

$$N_p(0,\Sigma)' \left[\frac{\text{Wishart}_{p,df}}{df}\right]^{-1} N_p(0,\Sigma) \sim \frac{(n-1)p}{n-p} F_{p,n-p},$$

where df is the degrees of freedom of the Wishart matrix, N and W are independent.

At the α level of significance, H_0 is rejected if the observed

$$T^2 = n(\bar{\mathbf{X}} - \mu_0)'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \mu_0) > \frac{(n-1)p}{n-p} F_{p,n-p}(\alpha).$$

Example 4.1 (Testing a multivariate mean vector with T^2) Perspiration from 20 healthy females was analyzed. Three components, X_1 =sweat rate, X_2 =sodium content, and X_3 =potassium content, were measured. The data are presented in the following table. Assume that $(X_1, X_2, X_3)'$ follows a multivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)'$.

Sweat Data

Individual	X_1 (Sweat rate)	X_2 (Sodium)	X_3 (Potassium)
1	3.7	48.5	9.3
2	5.7	65.1	8.0
3	3.8	47.2	10.9
4	3.2	53.2	12.0
5	3.1	55.5	9.7
6	4.6	36.1	7.9
7	2.4	24.8	14.0
8	7.2	33.1	7.6
9	6.7	47.4	8.5
10	5.4	54.1	11.3
11	3.9	36.9	12.7
12	4.5	58.8	12.3
13	3.5	27.8	9.8
14	4.5	40.2	8.4
15	1.5	13.5	10.1
16	8.5	56.4	7.1
17	4.5	71.6	8.2
18	6.5	52.8	10.9
19	4.1	44.1	11.2
20	5.5	40.9	9.4

Source: Courtesy of Dr. Gerald Bargman.

Statistical Hypothesis:

$$H_0 : \mu' = [4, 50, 10] \quad \text{versus} \quad H_1 : \mu' \neq [4, 50, 10].$$

Solution:

$$\bar{\mathbf{x}} = \begin{bmatrix} 4.640 \\ 45.400 \\ 9.965 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 2.879 & 10.010 & -1.810 \\ 10.010 & 199.788 & -5.640 \\ -1.810 & -5.640 & 3.628 \end{bmatrix} \quad \mathbf{S}^{-1} = \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix}$$

$$T^2 = 20[4.640-4, \ 45.400-50, \ 9.965-10] \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix} \begin{bmatrix} 4.640-4 \\ 45.400-50 \\ 9.965-10 \end{bmatrix}$$

$$= 20[.640, \ -4.600, \ -.035] \begin{bmatrix} .467 \\ -.042 \\ .160 \end{bmatrix} = 9.74$$

Comparing the observed $T^2 = 9.74$ with the critical value

$$\frac{(n-1)p}{(n-p)} F_{p,n-p}(.10) = \frac{19(3)}{17} F_{3,17}(.10) = 3.353(2.44) = 8.18$$

we see that $T^2 = 9.74 > 8.18$, and consequently, we reject H_0 at the 10% level of significance.

We note that H_0 will be rejected if one or more of the component means, or some combination of means, differs too much from the hypothesized values [4, 50, 10]. At this point, we have no idea which of these hypothesized values may not be supported by the data.

We have assumed that the sweat data are multivariate normal. The Q–Q plots constructed from the marginal distributions of X_1, X_2, and X_3 all approximate straight lines. Moreover, scatter plots for pairs of observations have approximate elliptical shapes, and we conclude that the normality assumption was reasonable in this case. (See Exercise 5.4.) ■

Hotelling's T^2 and Likelihood Ratio Tests

T^2-statistic can be derived as the likelihood ratio test of H_0.

- The maximum of the multivariate normal likelihood as μ and Σ are varied over their possible values is given by

$$\max_{\mu,\Sigma} L(\mu,\Sigma) = \frac{1}{(2\pi e)^{np/2}|\hat{\Sigma}|^{n/2}}, \quad \hat{\Sigma} = S_n.$$

- The maximum of the multivariate normal likelihood as $\mu = \mu_0$ and Σ is varied over their possible values is given by

$$\max_{\Sigma} L(\mu_0,\Sigma) = \frac{1}{(2\pi e)^{np/2}|\hat{\Sigma}_0|^{n/2}}$$

where $\hat{\Sigma} = n^{-1}\sum_{j=1}^{n}(\mathbf{x}_j - \mu_0)(\mathbf{x}_j - \mu_0)'$.

- Likelihood ratio

$$\Lambda = \frac{\max_{\Sigma} L(\mu_0,\Sigma)}{\max_{\mu,\Sigma} L(\mu,\Sigma)} = \left(\frac{\hat{\Sigma}}{\hat{\Sigma}_0}\right)^{n/2}.$$

$\Lambda^{2/n} = |\hat{\Sigma}|/|\hat{\Sigma}_0|$ is called Wilks' lambda.

Smaller value of Λ indicates an evidence against H_0.

Equivalence between Hotelling's T and Wilks' Λ

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample from an $N_p(\mu, \Sigma)$ population. Then the test based on T^2 is equivalent to the likelihood ratio test because

$$\Lambda^{2/n} = \left(1 + \frac{T^2}{n-1}\right)^{-1}.$$

Confidence Region

Question of Interest: Let θ be a vector of unknown population parameters. We want to find a confidence region which is a region of likely θ values.

The region $R(\mathbf{X})$, determined by the data, is said to be a $100(1-\alpha)\%$ confidence region if, before the sample is selected,

$$P(R(X) \text{ will cover the true } \theta) = 1 - \alpha.$$

This probability is calculated under the true, but unknown, value of θ.

Confidence Region of μ

A $100(1-\alpha)\%$ *confidence region* for the mean of a p-dimensional normal distribution is the ellipsoid determined by all $\boldsymbol{\mu}$ such that

$$n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}) \le \frac{p(n-1)}{(n-p)} F_{p,n-p}(\alpha)$$

where $\bar{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j$, $\mathbf{S} = \frac{1}{(n-1)} \sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the sample observations.

Confidence Region of μ

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$$n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) \le \frac{p(n-1)}{(n-p)} F_{p,n-p}(\alpha)$$

where $\bar{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j$, $\mathbf{S} = \frac{1}{(n-1)}\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the sample observations.

Geometry of the confidence interval: The joint confidence region for μ is an ellipsoid centered at $\bar{\mathbf{x}}$, and with axes

$$\pm\sqrt{\frac{\lambda_i p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)}\mathbf{e}_i, \quad \text{where } \mathbf{S}\mathbf{e}_i = \lambda_i\mathbf{e}_i, \quad , i = 1, 2, \ldots, p.$$

Simultaneous Confidence Statement

Let $\mathbf{X}$ have an $N_p(\mu, \Sigma)$ distribution and consider the linear combination

$$Z = \mathbf{a}'\mathbf{X} = a_1 X_1 + z_2 X_2 + \cdots + a_p X_p.$$

Note that $Z \sim N(\mathbf{a}'\mu, \mathbf{a}'\Sigma\mathbf{a})$. So, for a particular $\mathbf{a}$, a $100(1-\alpha)\%$ confidence interval of $\mathbf{a}'\mu$ is

$$\bar{z} - t_{n-1}(\alpha/2)\frac{s_z}{\sqrt{n}} \le \mathbf{a}'\mu \le \bar{z} + t_{n-1}(\alpha/2)\frac{s_z}{\sqrt{n}}$$

or

$$\mathbf{a}'\bar{\mathbf{x}} - t_{n-1}(\alpha/2)\frac{\sqrt{\mathbf{a}'S\mathbf{a}}}{\sqrt{n}} \le \mathbf{a}'\mu \le \mathbf{a}'\bar{\mathbf{x}} + t_{n-1}(\alpha/2)\frac{\sqrt{\mathbf{a}'S\mathbf{a}}}{\sqrt{n}}. \qquad \textbf{(CI)}$$

Question: For a fixed $\mathbf{a}$, the probability of (CI) contains the true $\mathbf{a}'\mu$ is $1-\alpha$.

- Is it true that the probability of (CI) containing the true value of $\mathbf{a}'\mu$, for all $\mathbf{a} \in \mathbb{R}^p$, is $1-\alpha$? (No!)
- How to construct a simultaneous confidence interval for all $\mathbf{a}'\mu$?

T^2 Confidence Intervals

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample from an $N_p(\mu, \Sigma)$. Then, simultaneously for all $\mathbf{a}$, the interval

$$\left(\mathbf{a}'\bar{\mathbf{X}} - \sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)\mathbf{a}'\mathbf{S}\mathbf{a}}, \quad \mathbf{a}'\bar{\mathbf{X}} + \sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)\mathbf{a}'\mathbf{S}\mathbf{a}}\right)$$

will contain $\mathbf{a}'\mu$ with probability $1-\alpha$.

Proof: Note that

$$\max_{\mathbf{a}} \frac{n[\mathbf{a}'(\bar{\mathbf{x}}-\mu)]^2}{\mathbf{a}'\mathbf{S}\mathbf{a}} = n(\bar{\mathbf{x}}-\mu)'\mathbf{S}^{-1}(\bar{\mathbf{x}}-\mu) = T^2.$$

SCIs for Linear Combinations

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample from an $N_p(\mu, \Sigma)$. Then, simultaneously for all $\mathbf{a}$, the interval

$$\left(\mathbf{a}'\bar{\mathbf{X}} - \sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)\mathbf{a}'\mathbf{S}\mathbf{a}}, \quad \mathbf{a}'\bar{\mathbf{X}} + \sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)\mathbf{a}'\mathbf{S}\mathbf{a}}\right)$$

will contain $\mathbf{a}'\mu$ with probability $1-\alpha$.

Let $\mathbf{a}' = [1, 0, \ldots, 0]$, $[0, 1, 0, \ldots, 0]$ and so on, we have

$$\begin{array}{c}
\bar{x}_1 - \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha)}\sqrt{\frac{s_{11}}{n}} \le \mu_1 \le \bar{x}_1 + \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha)}\sqrt{\frac{s_{11}}{n}} \\
\bar{x}_2 - \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha)}\sqrt{\frac{s_{22}}{n}} \le \mu_2 \le \bar{x}_2 + \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha)}\sqrt{\frac{s_{22}}{n}} \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\bar{x}_p - \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha)}\sqrt{\frac{s_{pp}}{n}} \le \mu_p \le \bar{x}_p + \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha)}\sqrt{\frac{s_{pp}}{n}}
\end{array}$$

all hold simultaneously with confidence coefficient at least $1-\alpha$.

Example 4.2 (Constructing simultaneous confidence intervals and ellipses). The scores obtained by $n = 87$ college students on the College Level Examination Program (CLEP) subtest X_1, and the College Qualification Test (CQT) subtests X_2 and X_3 are given in the following table for X_1 =social science and history, X_2 =verbal, and X_3=science.

College Test Data

Individual	X_1 (Social science and history)	X_2 (Verbal)	X_3 (Science)	Individual	X_1 (Social science and history)	X_2 (Verbal)	X_3 (Science)
1	468	41	26	45	494	41	24
2	428	39	26	46	541	47	25
3	514	53	21	47	362	36	17
4	547	67	33	48	408	28	17
5	614	61	27	49	594	68	23

Remark:

- The simultaneous T^2 confidence intervals are ideal for "data snooping".
- We can conclude the statement that (μ_i, μ_k) belong to the sample mean-centered ellipses

$$n[\bar{x}_i - \mu_i, \bar{x}_k - \mu_k] \begin{bmatrix} s_{ii} & s_{ik} \\ s_{ik} & s_{kk} \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_i - \mu_i \\ \bar{x}_k - \mu_k \end{bmatrix} \leq \frac{p(n-1)}{n-p} F_{p,n-p}(\alpha)$$

  and still maintain the confidence coefficient $(1-\alpha)$ for the whole set of statements.
- The simultaneous T^2 confidence intervals for the individual components of a mean vector are just the shadows, or projections, of the confidence ellipsoid on the component axes.

Bonferroni Simultaneous Confidence Intervals

Why Bonferroni?

When the number of individual confidence statements is small, it is possible to do better than the Scheffe's simultaneous confidence intervals.

Suppose that, prior to the collection of data, confidence statements about m linear combinations $\mathbf{a}_1'\mu, \dots, \mathbf{a}_m'\mu$ are required.

Recall, for any particular $\mathbf{a}_i'\mu$, a $1-\alpha$ confidence interval is given by

$$\mathbf{a}_i'\bar{\mathbf{x}} \pm t_{n-1}\left(\frac{\alpha}{2}\right)\sqrt{\frac{\mathbf{a}_i'S\mathbf{a}_i}{n}}, \quad i = 1, 2, \dots, m.$$

Bonferroni SCI replaces $t_{n-1}\left(\frac{\alpha}{2}\right)$ by $t_{n-1}\left(\frac{\alpha}{2m}\right)$. That is

$$\mathbf{a}_i'\bar{\mathbf{x}} \pm t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{\mathbf{a}_i'S\mathbf{a}_i}{n}}, \quad i = 1, 2, \dots, m.$$

When the sample size is large, tests of hypotheses and confidence regions for μ can be constructed without the assumption of a normal population.

Test of Mean Vector Based on Large Sample Theory

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be random sample from a population with mean μ and positive definite covariance matrix Σ. When $n - p$ is large, the hypothesis $H_0 : \mu = \mu_0$ is rejected in favor of $H_1 : \mu \neq \mu_0$, at a level of significance approximately α, if the observed

$$n(\bar{\mathbf{x}} - \mu_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \mu_0) > \chi_p^2(\alpha),$$

where $\chi_p^2(\alpha)$ is the upper 100α-th percentile χ_p^2.

Confidence Regions Based on Large Sample Theory

Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be random sample from a population with mean μ and positive definite covariance matrix Σ. When $n - p$ is large,

$$\mathbf{a}'\bar{\mathbf{x}} \pm \sqrt{\frac{\chi_p^2(\alpha)\mathbf{a}'\mathbf{S}\mathbf{a}}{n}}$$

will contain $\mathbf{a}'\mu$, for every $\mathbf{a}$, with probability approximately $1 - \alpha$. Consequently, we can make the $200(1 - \alpha)\%$ simultaneous confidence statements

$$\bar{x}_1 \pm \sqrt{\frac{\chi_p^2(\alpha)s_{11}}{n}} \text{ contains } \mu_1, \quad \dots, \quad \bar{x}_p \pm \sqrt{\frac{\chi_p^2(\alpha)s_{pp}}{n}} \text{ contains } \mu_p,$$

and, in addition, for all pairs (μ_i, μ_k), $i, k = 1, 2, \dots, p$ the sample mean-centered ellipses

$$n[\bar{x}_i - \mu_i, \bar{x}_k - \mu_k] \begin{bmatrix} s_{ii} & s_{ik} \\ s_{ik} & s_{kk} \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_i - \mu_i \\ \bar{x}_k - \mu_k \end{bmatrix} \le \chi_p^2(\alpha) \text{ contains } (\mu_i, \mu_k).$$

When the sample size is large, the one-at-a-time confidence intervals for individual means are

$$\bar{x}_i \pm z(\alpha/2)\sqrt{\frac{s_{ii}}{n}}, \quad i = 1, 2, \ldots, p,$$

where $z(\alpha/2)$ is the upper $100\alpha/2$-th percentile of the standard normal distribution.

The Bonferroni SCIs for the $m = p$ statements about the individual means are

$$\bar{x}_i \pm z(\alpha/(2p))\sqrt{\frac{s_{ii}}{n}}, \quad i = 1, 2, \ldots, p.$$

Outline

Topics in the part:

In this part, we shall introduce procedures for compare several mean vectors by assuming multivariate normal distribution or large sample sizes.

We shall often review univariate procedures for comparing several means and then generalize to the corresponding multivariate cases by analogy.

Efficacy of a new drug may be determined by comparing measurements with the "treatment" and without the "treatment".

One ideal approach is to assign both treatments to the same or identical units. The paired responses are analyzed by computing the their differences, thereby eliminating much of the influence of extraneous unit-to-unit variation.

What we did in univariate response case?

Data:

Obs.	Trt. 1	Trt. 2
1	X_{11}	X_{12}
2	X_{21}	X_{22}
$\vdots$	$\vdots$	$\vdots$
n	X_{n1}	X_{n2}

Given that the differences $D_j = X_{j1} - X_{j2}, j = 1, 2, \ldots, n$ are i.i.d. observations from $N(\delta, \sigma_\delta^2)$, we have

$$t = \frac{\bar{D} - \delta}{s_d/\sqrt{n}} \sim t_{n-1},$$

where $\bar{D}$ and s_d are the sample mean and sample standard deviation of D_j's.

Multivariate Case

Data:

Obs.	Trt. 1	Trt. 2
1	$(X_{111}, X_{112}, \ldots, X_{11p})$	$(X_{211}, X_{212}, \ldots, X_{21p})$
2	$(X_{121}, X_{122}, \ldots, X_{12p})$	$(X_{221}, X_{222}, \ldots, X_{22p})$
$\ldots$	$\ldots$	$\ldots$
j	$(X_{1j1}, X_{1j2}, \ldots, X_{1jp})$	$(X_{2j1}, X_{2j2}, \ldots, X_{2jp})$
$\ldots$	$\ldots$	$\ldots$
n	$(X_{1n1}, X_{1n2}, \ldots, X_{1np})$	$(X_{2n1}, X_{2n2}, \ldots, X_{2np})$

Let

$$D_{j1} = X_{1j1} - X_{2j1}, D_{j2} = X_{1j2} - X_{2j2}, \ldots, D_{jp} = X_{1jp} - X_{2jp}.$$

Define $\mathbf{D}_j' = [D_{j1}, D_{j2}, \ldots, D_{jp}]$. We assume that $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_n$ are i.i.d. $N_p(\delta, \Sigma_d)$.

Inferences about the vector of mean differences δ can be based upon a T^2-statistic

$$T^2 = n(\bar{\mathbf{D}} - \delta)' S_d^{-1} (\bar{\mathbf{D}} - \delta).$$

Distribution of T^2

Let the differences $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_n$ be a random sample from an $N_p(\delta, \Sigma_d)$. Then

$$T^2 = n(\bar{\mathbf{D}} - \delta)' S_d^{-1} (\bar{\mathbf{D}} - \delta) \sim \frac{(n-1)pF_{p,n-p}}{n-p}.$$

Hypothesis Testing: An α-level test of

$$H_0 : \delta = 0 \quad \text{versus} \quad H_1 : \delta \neq 0$$

rejects H_0 if the observed

$$T^2 = n\bar{\mathbf{d}}' S_d^{-1} \bar{\mathbf{d}} > \frac{(n-1)pF_{p,n-p}(\alpha)}{n-p}$$

Confidence Interval: A $100(1-\alpha)\%$ confidence region for δ

$$n(\bar{\mathbf{d}} - \delta)' S_d^{-1} (\bar{\mathbf{d}} - \delta) \leq \frac{(n-1)pF_{p,n-p}(\alpha)}{n-p}.$$

Simultaneous Confidence Interval

A $100(1-\alpha)\%$ simultaneous confidence intervals for the individual mean differences δ_i are given by

$$\bar{d}_i \pm \sqrt{\frac{(n-1)pF_{p,n-p}(\alpha)s_{d_i}^2}{n(n-p)}},$$

where $\bar{d}_i$ is the i-th element of $\bar{\mathbf{d}}$ and $s_{d_i}^2$ is the i-th diagonal element of $\mathbf{S}_d$.

Bonferroni Simultaneous Confidence Interval

A $100(1-\alpha)\%$ Bonferroni simultaneous confidence intervals for the individual mean differences δ_i are given by

$$\bar{d}_i \pm t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{s_{d_i}^2}{n}}.$$

Simultaneous Confidence Interval: Non-Normal Data: If $n-p$ is large, replace

$$\frac{(n-1)pF_{p,n-p}(\alpha)}{n-p} \quad \text{with} \quad \chi_p^2(\alpha).$$

Example 4.3 (Checking for a mean difference with paired observations) Municipal wastewater treatment plants are required by law to monitor their discharges into rivers and streams on a regular basis. Concern about the reliability of data from one of these self-monitoring programs led to a study in which samples of effluent were divided and sent to two laboratories for testing. One-half of each sample was sent to the Wisconsin State Laboratory of Hygiene, and one-half was sent to a private commercial laboratory routinely used in the monitoring program. Measurements of biochemical oxygen demand (BOD) and suspended solids (SS) were obtained, for $n = 11$ sample splits, from the two laboratories.

Effluent Data

Sample j	Commercial lab x_{1j1} (BOD)	Commercial lab x_{1j2} (SS)	State lab of hygiene x_{2j1} (BOD)	State lab of hygiene x_{2j2} (SS)
1	6	27	25	15
2	6	23	28	13
3	18	64	36	22
4	8	44	35	29
5	11	30	15	31
6	34	75	44	64
7	28	26	42	30
8	71	124	54	64
9	43	54	34	56
10	33	30	29	20
11	20	14	39	21

Consider a situation where q treatments are compared with respect to a single response variable. Each subject or experimental unit receives each treatment once over successive periods of time.

Data:

Obs.	Trt.1	Trt.2	$\cdots$	Trt.q	
1	X_{11}	X_{12}	$\ldots$	X_{1q}	$\mathbf{X}_1'$
2	X_{21}	X_{22}	$\ldots$	X_{2q}	$\mathbf{X}_2'$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
n	X_{n1}	X_{n2}	$\ldots$	X_{nq}	$\mathbf{X}_n'$

Model: Assume that $\mathbf{X}_j$'s are i.i.d. with $N_q(\mu, \Sigma)$.

Statistical Inference: Test of Equality of Treatments.

The null hypothesis of testing the equality of treatment means can be formulated as

$$\begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \\ \vdots \\ \mu_1 - \mu_q \end{bmatrix} = \mathbf{0}, \quad \text{or} \quad \begin{bmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_2 \\ \vdots \\ \mu_q - \mu_{q-1} \end{bmatrix} = \mathbf{0}.$$

If we define

$$C_1 = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix},$$

the testing the equality of treatment means is equivalent to testing $C_1\mu = 0$ or $C_2\mu = 0$.

C_1, C_2 are called <u>contrast matrix</u>.

Test for Equality of Treatments

Consider an $N_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ population, and let $\mathbf{C}$ be a contrast matrix. An α-level test of H_0: $\mathbf{C}\boldsymbol{\mu} = \mathbf{0}$ (equal treatment means) versus H_1: $\mathbf{C}\boldsymbol{\mu} \neq \mathbf{0}$ is as follows: Reject H_0 if

$$T^2 = n(\mathbf{C}\bar{\mathbf{x}})'(\mathbf{CSC}')^{-1}\mathbf{C}\bar{\mathbf{x}} > \frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha) \qquad (*)$$

where $F_{q-1,n-q+1}(\alpha)$ is the upper (100α)th percentile of an F-distribution with $q-1$ and $n-q+1$ d.f. Here $\bar{\mathbf{x}}$ and $\mathbf{S}$ are the sample mean vector and covariance matrix defined, respectively, by

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j \quad \text{and} \quad \mathbf{S} = \frac{1}{n-1}\sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$$

Note: T^2 does not depend on the particular choice of C.

Confidence Region for Constrasts

A confidence region for contrasts $\mathbf{C}\boldsymbol{\mu}$, with $\boldsymbol{\mu}$ the mean of a normal population, is determined by the set of all $\mathbf{C}\boldsymbol{\mu}$ such that

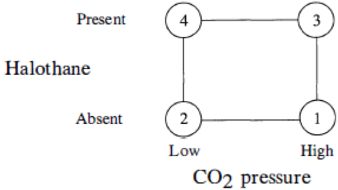
$$n(\mathbf{C}\bar{\mathbf{x}} - \mathbf{C}\boldsymbol{\mu})'(\mathbf{CSC}')^{-1}(\mathbf{C}\bar{\mathbf{x}} - \mathbf{C}\boldsymbol{\mu}) \leq \frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,\,n-q+1}(\alpha)$$

where $\bar{\mathbf{x}}$ and $\mathbf{S}$ are as defined in (*). Consequently, simultaneous $100(1-\alpha)\%$ confidence intervals for single contrasts $\mathbf{c}'\boldsymbol{\mu}$ for any contrast vectors of interest are given by

$$\mathbf{c}'\boldsymbol{\mu}: \quad \mathbf{c}'\bar{\mathbf{x}} \pm \sqrt{\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha)} \sqrt{\frac{\mathbf{c}'\mathbf{S}\mathbf{c}}{n}}$$

Example 4.4 (Testing for equal treatments in a repeated measure design)

Improved anesthetics are often developed by first studying their effects on animals. In one study, 19 dogs were initially given the drug pentobarbitol. Each dog was then administered carbon dioxide (CO_2) at each of two pressure levels. Next, halothane (H) was added, and the administration of (CO_2) was repeated. The response, milliseconds between heartbeats, was measured for the four treatment combinations:



Treatment 1 = high CO_2 pressure without H

Treatment 2 = low CO_2 pressure without H

Treatment 3 = high CO_2 pressure with H

Treatment 4 = low CO_2 pressure with H

Let μ_1, μ_2, μ_3 and μ_4 correspond to the mean responses for treatment 1,2,3, and 4, respectively.

Consider the following three treatment contrasts:

$$
\begin{aligned}
(\mu_3 + \mu_4) - (\mu_1 + \mu_2) &: \quad \text{Halothane effect} \\
(\mu_1 + \mu_3) - (\mu_2 + \mu_4) &: \quad \text{CO}_2 \text{ effect} \\
(\mu_1 + \mu_4) - (\mu_2 + \mu_3) &: \quad \text{Interaction effect}
\end{aligned}
$$

Test all three effects are 0. That is,

$$H_0 : C\mu = 0 \quad \text{versus} \quad H_1 : C\mu \neq 0,$$

where

$$C = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Data Structure

Suppose we have two populations $\mathbf{X}_1$ and $\mathbf{X}_2$.

Assumptions Concerning the Structure of the Data:

- The sample $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$ is from $\mathbf{X}_1$;
- The sample $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$ is from $\mathbf{X}_2$;
- These two samples are independent.

Normal Populations with Equal Covariance

We further assume that

- $\mathbf{X}_1 \sim N_p(\boldsymbol{\mu}_1, \Sigma_1)$, $\mathbf{X}_2 \sim N_p(\boldsymbol{\mu}_2, \Sigma_2)$,
- $\Sigma_1 = \Sigma_2 = \Sigma$.

Note that

- $\boldsymbol{\mu}_1$ can be estimated by $\bar{X}_1$, the sample mean of $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$,
- $\boldsymbol{\mu}_2$ can be estimated by $\bar{X}_1$, the sample mean of $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$,
- Σ can be estimated by the pooled covariance matrix

$$\mathbf{S}_{\text{pooled}} = \frac{(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2}{n_1 + n_2 - 2},$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the sample covariance of these two samples, respectively.

Test: Two Populations

If $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$ is a random sample of size n_1 from $N_p(\boldsymbol{\mu}_1, \Sigma)$, and $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$ is a random sample of size n_2 from $N_p(\boldsymbol{\mu}_2, \Sigma)$, then

$$T^2 = [(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2) - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)]' \left[\left(\frac{1}{n_1} + \frac{1}{n_2} \right) S_{\text{pooled}} \right]^{-1} [(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2) - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)]$$

is distributed as

$$\frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1+n_2-p-1}.$$

Hypothesis Testing: Reject $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ and in favor of $H_1 : \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$ at the significance level α if

$$(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2} \right) S_{\text{pooled}} \right]^{-1} (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2) \geq \frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1+n_2-p-1}(\alpha).$$

Confidence Region: A $100(1-\alpha)\%$ confidence region of $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$ is given by

$$T^2 \leq \frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1+n_2-p-1}(\alpha).$$

Example (Constructing a confidence region for the difference of two mean vectors)

Fifty bars of soap are manufactured in each of two ways. Two characteristics, X_1 = lather and X_2 = mildness, are measured. The summary statistics for bars produced by methods 1 and 2 are

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 8.3 \\ 4.1 \end{bmatrix}, \qquad \mathbf{S}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} 10.2 \\ 3.9 \end{bmatrix}, \qquad \mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Obtain a 95% confidence region for $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$.

$$\mathbf{S}_{\text{pooled}} = \frac{49}{98}\mathbf{S}_1 + \frac{49}{98}\mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \qquad \bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 = \begin{bmatrix} -1.9 \\ .2 \end{bmatrix}$$

so the confidence ellipse is centered at $[-1.9, .2]'$. The eigenvalues and eigenvectors of $\mathbf{S}_{\text{pooled}}$ are obtained from the equation

$$0 = |\mathbf{S}_{\text{pooled}} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 9$$

so $\lambda = (7 \pm \sqrt{49-36})/2$. Consequently, $\lambda_1 = 5.303$ and $\lambda_2 = 1.697$, and the corresponding eigenvectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, determined from

$$\mathbf{S}_{\text{pooled}}\,\mathbf{e}_i = \lambda_i \mathbf{e}_i, \qquad i = 1, 2$$

are

$$\mathbf{e}_1 = \begin{bmatrix} .290 \\ .957 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} .957 \\ -.290 \end{bmatrix}$$

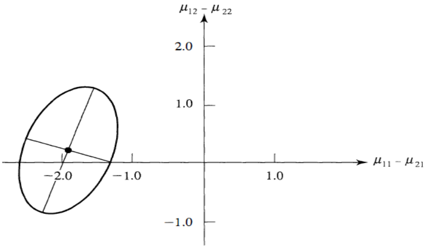
By Result 6.2,

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right)c^2 = \left(\frac{1}{50} + \frac{1}{50}\right)\frac{(98)(2)}{(97)} F_{2,97}(.05) = .25$$

since $F_{2,97}(.05) = 3.1$. The confidence ellipse extends

$$\sqrt{\lambda_i}\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)c^2} = \sqrt{\lambda_i}\,\sqrt{.25}$$

units along the eigenvector $\mathbf{e}_i$, or 1.15 units in the $\mathbf{e}_1$ direction and .65 units in the $\mathbf{e}_2$ direction.



95% confidence ellipse for $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$.

Suppose we have samples from g populations

Population 1: $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$;

Population 2: $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$;

$\cdots, \cdots$;

Population g: $\mathbf{X}_{g1}, \mathbf{X}_{g2}, \ldots, \mathbf{X}_{gn_g}$.

Assumptions:

- The random samples from different populations are independent;
- All populations have a common covariance matrix Σ;
- The k-th population has a normal distribution: $N_p(\mu_k, \Sigma)$, $j = 1, \ldots, g$.

Hypothesis to test:

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \cdots = \boldsymbol{\mu}_g \quad \text{versus} \quad H_1 : H_0 \text{ is not true.}$$

MANOVA Statistical Model

We shall assume the data are from the following model

$$\mathbf{X}_{lj} = \boldsymbol{\mu}_1 + \boldsymbol{\tau}_l + \mathbf{e}_{lj}, \quad j = 1, 2, \ldots, n_l \text{ and } l = 1, 2, \ldots, g$$

where the $\mathbf{e}_{lj}$ are independent $N_p(0, \Sigma)$ variables. Here the parameter vector $\boldsymbol{\mu}$ is the overall mean, and $\boldsymbol{\tau}_l$ represents the l-th treatment effect with $\sum_{l=1}^{g} n_l \boldsymbol{\tau}_l = 0$.

Accordingly, a vector of observation may be decomposed as

$$\mathbf{x}_{lj} = \bar{\mathbf{x}} + (\bar{\mathbf{x}}_l - \bar{\mathbf{x}}) + (\mathbf{x}_{lj} - \bar{\mathbf{x}}_l).$$

Then we have the following decomposition of sample covariances

$$\sum_{l=1}^{g}\sum_{j=1}^{n_l}(\mathbf{x}_{lj} - \bar{\mathbf{x}})(\mathbf{x}_{lj} - \bar{\mathbf{x}})' = \sum_{l=1}^{g} n_l(\bar{\mathbf{x}}_l - \bar{\mathbf{x}})(\bar{\mathbf{x}}_l - \bar{\mathbf{x}})' + \sum_{l=1}^{g}\sum_{j=1}^{n_l}(\mathbf{x}_{lj} - \bar{\mathbf{x}}_l)(\mathbf{x}_{lj} - \bar{\mathbf{x}}_l)'$$

Total SS = Treatment (**B**etween) SS + Residual (**W**ithin) SS

MANOVA Table

MANOVA TABLE FOR COMPARING POPULATION MEAN VECTORS

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Treatment	$\mathbf{B} = \sum_{\ell=1}^{g} n_\ell(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})'$	$g - 1$
Residual (Error)	$\mathbf{W} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)'$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$\mathbf{B} + \mathbf{W} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}})(\mathbf{x}_{\ell j} - \bar{\mathbf{x}})'$	$\sum_{\ell=1}^{g} n_\ell - 1$

The hypothesis H_0 will be rejected if

$$\Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|} = \frac{\left|\sum_{l=1}^{g}\sum_{j=1}^{n_l}(\mathbf{x}_{lj}-\bar{\mathbf{x}}_l)(\mathbf{x}_{lj}-\bar{\mathbf{x}}_l)'\right|}{\left|\sum_{l=1}^{g}\sum_{j=1}^{n_l}(\mathbf{x}_{lj}-\bar{\mathbf{x}})(\mathbf{x}_{lj}-\bar{\mathbf{x}})'\right|}$$

is too small.

Distribution of Wilk's Lambda

Special Cases

DISTRIBUTION OF WILKS' LAMBDA, $\Lambda^* = |\mathbf{W}|/|\mathbf{B}+\mathbf{W}|$

No. of variables	No. of groups	Sampling distribution for multivariate normal data
$p = 1$	$g \geq 2$	$\left(\frac{\Sigma n_\ell - g}{g-1}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{g-1, \Sigma n_\ell - g}$
$p = 2$	$g \geq 2$	$\left(\frac{\Sigma n_\ell - g - 1}{g-1}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(g-1), 2(\Sigma n_\ell - g - 1)}$
$p \geq 1$	$g = 2$	$\left(\frac{\Sigma n_\ell - p - 1}{p}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{p, \Sigma n_\ell - p - 1}$
$p \geq 1$	$g = 3$	$\left(\frac{\Sigma n_\ell - p - 2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2p, 2(\Sigma n_\ell - p - 2)}$

Large Sample Cases: Bartlett has shown that if H_0 is true and $\sum_{l=1}^{g} n_l = n$ is large, then

$$-\left(n - 1 - \frac{p+q}{2}\right)\ln\Lambda^* \Longrightarrow \chi^2_{p(q-1)}.$$

Consequently, we reject H_0 at the significance level α if

$$-\left(n - 1 - \frac{p+q}{2}\right)\ln\Lambda^* > \chi^2_{p(q-1)}(\alpha).$$

Example 4.5 (A multivariate analysis of Wisconsin nursing home data)
The Wisconsin Department of Health and Social Services reimburses nursing homes in the state for the services provided. The department develops a set of formulas for rates for each facility, based on factors such as level of care, mean wage rate, and average wage rate in the state.

Nursing homes can be classified on the basis of ownership (private party, nonprofit organization, and government) and certification (skilled nursing facility, intermediate care facility, or a combination of the two).

One purpose of a recent study was to investigate the effects of ownership or certification (or both) on costs. Four costs, computed on a per-patient-day basis and measured in hours per patient day, were selected for analysis: X_1 =cost of nursing labor, X_2 =cost of dietary labor, X_3 =cost of plant operation and maintenance labor, and X_4 =cost of housekeeping and laundry labor. A total of $n = 516$ observations on each of the $p = 4$ cost variables were initially separated according to ownership. Summary statistics for each of the $g = 3$ groups are given in the following table.

Group	Number of observations	Sample mean vectors
$\ell = 1$ (private)	$n_1 = 271$	
$\ell = 2$ (nonprofit)	$n_2 = 138$	$\bar{\mathbf{x}}_1 = \begin{bmatrix} 2.066 \\ .480 \\ .082 \\ .360 \end{bmatrix}; \quad \bar{\mathbf{x}}_2 = \begin{bmatrix} 2.167 \\ .596 \\ .124 \\ .418 \end{bmatrix}; \quad \bar{\mathbf{x}}_3 = \begin{bmatrix} 2.273 \\ .521 \\ .125 \\ .383 \end{bmatrix}$
$\ell = 3$ (government)	$n_3 = 107$	
	$\sum_{\ell=1}^{3} n_\ell = 516$	

$$\mathbf{S}_1 = \begin{bmatrix} .291 & & & \\ -.001 & .011 & & \\ .002 & .000 & .001 & \\ .010 & .003 & .000 & .010 \end{bmatrix}; \quad \mathbf{S}_2 = \begin{bmatrix} .561 & & & \\ .011 & .025 & & \\ .001 & .004 & .005 & \\ .037 & .007 & .002 & .019 \end{bmatrix};$$

$$\mathbf{S}_3 = \begin{bmatrix} .261 & & & \\ .030 & .017 & & \\ .003 & -.000 & .004 & \\ .018 & .006 & .001 & .013 \end{bmatrix}$$

Example 4.6 The relationship of size and shape for painted turtles is studied. The following table contains their measurements on the carapaces of 24 female and 24 male turtles. Test for equality of the two population mean vectors using $\alpha = 0.05$.

CARAPACE MEASUREMENTS (IN MILLIMETERS) FOR PAINTED TURTLES

Female			Male		
Length (x_1)	Width (x_2)	Height (x_3)	Length (x_1)	Width (x_2)	Height (x_3)
98	81	38	93	74	37
103	84	38	94	78	35
103	86	42	96	80	35
105	86	42	101	84	39
109	88	44	102	85	38
123	92	50	103	81	37
123	95	46	104	83	39
133	99	51	106	83	39

SCIs for Treatment Effects

When the hypothesis of equal treatment effects is rejected, those effects that led to the rejection of the hypothesis are of interest.

For pairwise comparisons, the Bonferroni approach will be used to construct the SCIs for the component differences $\boldsymbol{\tau}_k - \boldsymbol{\tau}_l$.

SCIs for Treatment Effects

Let $n = \sum_{k=1}^{g} n_k$. For the model in (6-38), with confidence at least $(1 - \alpha)$,

$$\tau_{ki} - \tau_{\ell i} \quad \text{belongs to} \quad \bar{x}_{ki} - \bar{x}_{\ell i} \pm t_{n-g}\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{w_{ii}}{n-g}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$$

for all components $i = 1, \ldots, p$ and all differences $\ell < k = 1, \ldots, g$. Here w_{ii} is the ith diagonal element of $\mathbf{W}$.

Outline

When comparing two or more multivariate mean vectors, we often assume that the covariance matrices of the potentially different populations are the same.

With g populations, the null hypothesis of interest is

$$H_0 : \Sigma_1 = \Sigma_2 = \cdots = \Sigma_g = \Sigma.$$

The alternative hypothesis is that at least two of the covariance matrices are not equal.

Likelihood Ratio Test Statistic:

$$\Lambda = \prod_{j=1}^{g} \left(\frac{|\mathbf{S}_j|}{|\mathbf{S}_{\text{pooled}}|} \right)^{(n_j-1)/2}$$

where

- $\mathbf{S}_j$: the sample covariance matrix from j-th group;
- $\mathbf{S}_{\text{pooled}}$: the pooled sample covariance matrix

$$\mathbf{S}_{\text{pooled}} = \frac{(n_1 - 1)\mathbf{S}_1 + \cdots + (n_g - 1)\mathbf{S}_g}{\sum_{j=1}^{g}(n_j - 1)}.$$

Box's M Test Statistic

Box's test is based on the χ^2 approximation to the sampling distribution of

$$M = -2\log\Lambda = \left[\sum_{j=1}^{g}(n_j - 1)\right]\log|\mathbf{S}_{\text{pooled}}| - \sum_{j=1}^{g}[(n_j - 1)\log|\mathbf{S}_j|].$$

Box's Test for Equality of Covariance Matrices

Set

$$u\left[\sum_{j=1}^{g}\frac{1}{n_j - 1} - \frac{1}{\sum_{j=1}^{g}(n_j - 1)}\right]\cdot\frac{2p^2 + 3p - 1}{6(p+1)(g-1)}.$$

Then

$$C = (1-u)M$$

has an approximate χ^2 distribution with degrees of freedom $v = 0.5p(p+1)(g-1)$. Thus, at significant level α, reject H_0 if $C > \chi^2_v(\alpha)$.