Supplement for

# Checking Adequacy of Variance Function in Nonparametric Regression Model

Jia Liang, Weixing Song*

*Department of Statistics, Kansas State University, Manhattan, KS 66506*

**Abstract**

In this paper, we propose a class of test procedures to check the adequacy of parametric forms of the variance function in regression models when the mean function is unknown. Replacing the unknown mean function with the Nadaraya-Watson estimator, the proposed test statistics are constructed from a modified minimum distance between a nonparametric estimator and a parametric estimator under the null hypothesis for the variance function. Large sample properties of the estimator of the parameters in the variance function are discussed, and the asymptotic distribution of the test statistics under the null hypothesis is established, as well as the consistency and the power under some local alternative hypotheses. Extensive simulation and comparison studies indicate that the finite sample performance of the proposed test procedure are satisfactory. Finally, the applications of the proposed test on two real data sets are conducted.

*Keywords:* Lack-of-Fit test, Variance function, Minimum distance, Consistency and Local Power
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## 1. Lemma 3.1 From Koul and Ni (2004)

To prove the consistency of the minimum distance estimators $\hat{\theta}_n^*$ and $\hat{\theta}_n$, we need Lemma 3.1 from Koul and Ni (2004). Let $L_2(G)$ denote a class of square-integrable real-valued functions on $\mathbb{R}^d$ w.r.t. $G$. Define

$$\rho(v_1, v_2) = \int_{\mathcal{C}} [v_1(x) - v_2(x)]^2 dG(x) \qquad v_1,\ v_2 \in L_2(G)$$

and for a parametric function $m_\theta(x) \in L_2(G)$, we also define a map

$$M(v) = \text{argmin}_{\theta\in\Theta}\rho(v, m_\theta), \qquad v \in L_2(G).$$

Then Koul and Ni (2004) shows that

*Corresponding author
*Email address:* weixing@ksu.edu (Weixing Song)

**Lemma 1.1.** *Let $m_\theta(x)$ be such that*

*(m1). For each $\theta$, $m_\theta(x)$ is a.s. continuous in $x$, w.r.t. integrating measure $G$.*

*(m2). The parametric family of models $m_{\theta_1}(x) = m_{\theta_2}(x)$ a.s. $(G)$, then $\theta_1 = \theta_2$.*

*(m3). There exists a positive continuous function $l$ on $\mathcal{C}$ and for some $\beta > 0$,*

$$|m_{\theta_2}(x) - m_{\theta_1}(x)| \leq \| \theta_2 - \theta_1 \|^\beta l(x), \quad for\ any\ \theta_1, \theta_2 \in \Theta, x \in \mathcal{C}.$$

*Then the following hold*

*(a). $M(v)$ always exists, $v \in L_2(G)$.*

*(b). If $M(v)$ is unique, then $M$ is continuous at $v$ in the sense that for any sequence of $\{v_n\} \in L_2(G)$ converging to $v$ in $L_2(G)$, $M(v_n) \to M(v)$, that is,*

$$\rho(v_n, v) \to 0 \Longrightarrow M(v_n) \to M(v) \quad as\ n \to \infty.$$

*(c). $M(m_\theta(\cdot)) = \theta$, uniquely for any $\theta \in \Theta$.*

## 2. Detailed Proofs of Main Results

In this section, we will present the proofs of all the main results from Section 3. For the sake of brevity, we denote $K_{hi}(x) = K_h(X_i - x), K_{hij} = K_h(X_i - X_j)$, $v_0(x) = v(x, \theta_0)$, $K_{h1,2}(x) = K_{h1}(x)K_{h2}(x)$.

*The proof of Theorem 3.1.* Define

$$\hat{v}_n(x) = \frac{1}{n\hat{f}_w^3(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - \hat{m}(X_i))^2 \hat{f}_h^2(X_i), \quad v_{\theta_0}(x) = v_0(x).$$

Then $T_n^*(\theta) = \rho(\hat{v}_n, v_{\theta_0})$, and $\hat{\theta}_n^* = M(\hat{v}_n)$. Based on Lemma 1.1, to show the consistency of $\hat{\theta}_n^*$, it suffices to show $T_n^*(\theta_0) \to 0$ as $n \to \infty$.

Adding and subtracting $m(X_i)$ from $Y_i - \hat{m}(X_i)$, $T_n^*(\theta_0)$ can be written as the sum of six terms, and the first three terms are

$$T_{n1}^*(\theta_0) = \int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}_w^3(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 \hat{f}_h^2(X_i) - v_0(x) \right\}^2 dG(x)$$

$$T_{n2}^* = \int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}_w^3(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))(m(X_i) - \hat{m}(X_i)) \hat{f}_h^2(X_i) \right\}^2 dG(x)$$

$$T_{n3}^* = \int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}_w^3(x)} \sum_{i=1}^n K_{hi}(x)(m(X_i) - \hat{m}(X_i))^2 \hat{f}_h^2(X_i) \right\}^2 dG(x)$$

and the last three terms, by Cauchy-Schwartz inequality, are bounded above by $\sqrt{T^*_{n1}(\theta_0)T^*_{n2}}$, $\sqrt{T^*_{n1}(\theta_0)T^*_{n3}}$, and $\sqrt{T^*_{n2}T^*_{n3}}$, respectively. Thus, to show $T^*_n(\theta_0) \to 0$ in probability, it is sufficient to show $T^*_{n1}(\theta_0)$, $T^*_{n2}$ and $T^*_{n3}$ converge to 0.

Adding and subtracting $f^2(X_i)$ from $\hat{f}^2_h(X_i)$, $T^*_{n1}(\theta_0)$ can be further written as a sum of three terms, and the first two terms are

$$T^*_{n11}(\theta_0) = \int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}^3_w(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 f^2(X_i) - v_0(x) \right\}^2 dG(x)$$

$$T^*_{n12} = \int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}^3_w(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 (\hat{f}^2_h(X_i) - f^2(X_i)) \right\}^2 dG(x),$$

and the last term, by Cauchy-Schwartz inequality, are bounded above by $\sqrt{T^*_{n11}(\theta_0)T^*_{n12}}$. Thus, to show $T^*_{n1}(\theta_0) \to 0$ in probability, it is sufficient to show $T^*_{n11}(\theta_0)$ and $T^*_{n12}$ converge to 0. Note that $T^*_{n11}(\theta_0)$ can be further bounded above by

$$2\int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}_w(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 f^2(X_i) - f^2(x)v_0(x) \right\}^2 \hat{f}^{-4}_w(x)dG(x)$$
$$+2\int_{\mathcal{C}} \left\{ [\hat{f}^2_w(x) - f^2(x)]v_0(x) \right\}^2 \hat{f}^{-4}_w(x)dG(x).$$

By treating $(Y - m(X))^2 f^2(X)$ as the response variable, similar to the proof of the Corollary 3.1 in Koul and Ni (2004), we can easily show that the first term on the right hand side of the above inequality is the order of $o_p(1)$. The details are omitted here for the sake of brevity. Denote the second term on the right hand side of the above inequality as $T^*_{n13}$.

Next we show $T^*_{n12} \to 0$ in probability. Recall the notation $\hat{\varphi}_w$, we have

$$\int_{\mathcal{C}} \left\{ \frac{1}{n\hat{f}^3_w(x)} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 (\hat{f}^2_h(X_i) - f^2(X_i)) \right\}^2 dG(x)$$
$$\leq \sup_{x\in\mathcal{C}} \frac{f^6(x)}{\hat{f}^6_w(x)} \cdot \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 (\hat{f}^2_h(X_i) - f^2(X_i)) \right\}^2 d\varphi(x).$$

Furthermore, we have

$$\int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)(Y_i - m(X_i))^2 (\hat{f}^2_h(X_i) - f^2(X_i)) \right\}^2 d\varphi(x)$$
$$\leq 2\int \left\{ \hat{f}^2_h(X_i)\varepsilon_i^2 v_0(X_i) \left(\hat{f}_h(X_i) - f(X_i)\right)^2 \right\}^2 d\varphi(x)$$
$$+8\int \left\{ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)\varepsilon_i^2 v_0(X_i) f(X_i) \left(\hat{f}_h(X_i) - f(X_i)\right) \right\}^2 d\varphi(x).$$

Denote the right hand side of the above inequality as $2A_{n1}+8A_{n2}$. For $A_{n1}$, we have

$$EA_{n1}=\frac{1}{n}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(\frac{1}{nh^d}K(0)+\frac{1}{n}\sum_{j=2}^n K_{h1j}-f(X_1)\right)^4 d\varphi(x)$$
$$+\frac{n-1}{n}\int EK_{h1,2}(x)v_0(X_1)v_0(X_2)\left(\hat{f}_h(X_1)-f(X_1)\right)^2\left(\hat{f}_h(X_2)-f(X_2)\right)^2 d\varphi(x).$$

For the first term on the right hand side, we can show that it is bounded above by

$$\frac{8K^4(0)}{n^5h^{4d}}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)d\varphi(x)+\frac{8}{n}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(\frac{1}{n}\sum_{j=2}^n K_{h1j}-f(X_1)\right)^4 d\varphi(x)$$
$$\le O\left(\frac{1}{n^5h^{5d}}\right)+\frac{8}{n}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(\frac{1}{n}\sum_{j=2}^n\left(K_{h1j}-E[K_{h12}|X_1]\right)\right)^4 d\varphi(x)$$
$$+\frac{8}{n}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(E[K_{h12}|X_1]-f(X_1)\right)^4 d\varphi(x)$$
$$\le O\left(\frac{1}{n^5h^{5d}}\right)+\frac{8(n-1)}{n^5}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(K_{h12}-E[K_{h12}|X_1]\right)^4 d\varphi(x)$$
$$+O\left(\frac{1}{n^3}\right)\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(K_{h12}-E[K_{h12}|X_1]\right)^2\left(K_{h13}-E[K_{h12}|X_1]\right)^2 d\varphi(x)$$
$$+\frac{8}{n}\int EK_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)\left(E[K_{h12}|X_1]-f(X_1)\right)^4 d\varphi(x)$$
$$=O\left(\frac{1}{n^3h^{3d}}\right)+O\left(\frac{1}{nh^{d-8}}\right).$$

For the second term in $EA_{n1}$, denote $L_{12h}(x)=K_{h1,2}(x)v_0(X_1)v_0(X_2)$ for the sake of brevity,

$$\int EL_{12h}(x)\left(\hat{f}_h(X_1)-f(X_1)\right)^2\left(\hat{f}_h(X_2)-f(X_2)\right)^2 d\varphi(x)$$
$$=\int EL_{12h}(x)\left(\frac{1}{nh^d}K(0)+\frac{1}{n}K_{h12}+\frac{1}{n}\sum_{j=3}^n K_{h1j}-f(X_1)\right)^2$$
$$\left(\frac{1}{nh^d}K(0)+\frac{1}{n}K_{h12}+\frac{1}{n}\sum_{j=1,3}^n K_{h2j}-f(X_2)\right)^2 d\varphi(x)$$
$$\le O\left(\frac{1}{n^4h^{4d}}\right)+O\left(\frac{1}{n^2h^{2d}}\right)\int EL_{12h}(x)\left(\frac{1}{n}\sum_{j=3}^n K_{h1j}-f(X_1)\right)^2 d\phi(x)$$
$$+4\int EL_{12h}(x)\left(\frac{1}{n}\sum_{j=3}^n K_{h1j}-f(X_1)\right)^2\left(\frac{1}{n}\sum_{j=1,3}^n K_{h2j}-f(X_2)\right)^2 d\varphi(x)$$

$$
\begin{aligned}
&\le O\left(\frac{1}{n^2h^{2d}}\right)\int EL_{12h}(x)\left(\frac{1}{n}\sum_{j=3}^{n}[K_{h1j}-E(K_{h1j}|X_1)]\right)^2 d\phi(x)+O\left(\frac{1}{n^4h^{4d}}\right)\\
&\quad +O\left(\frac{1}{n^2h^{2d}}\right)\int EL_{12h}(x)\left(\frac{n-2}{n}E(K_{h1j|X_1})-f(X_1)]\right)^2 d\phi(x)\\
&\quad +16\int EL_{12h}(x)\left(\frac{1}{n}\sum_{j=3}^{n}[K_{h1j}-E(K_{h1j}|X_1)]\right)^2\left(\frac{1}{n}\sum_{j=1,3}^{n}[K_{h2j}-E(K_{h2j}|X_2)]\right)^2 d\varphi(x)\\
&\quad +16\int EL_{12h}(x)\left(\frac{n-2}{n}E(K_{h1j}|X_1)-f(X_1)\right)^2\left(\frac{1}{n}\sum_{j=1,3}^{n}[K_{h2j}-E(K_{h2j}|X_2)]\right)^2 d\varphi(x)\\
&\quad +16\int EL_{12h}(x)\left(\frac{n-2}{n}E(K_{h1j}|X_1)-f(X_1)\right)^2\left(\frac{n-2}{n}E(K_{h2j}|X_2)-f(X_2)\right)^2 d\varphi(x)\\
&= O\left(\frac{1}{n^4h^{4d}}\right)+O\left(\frac{1}{n^2h^{2d-8}}\right)+O(h^8).
\end{aligned}
$$

Therefore,

$$
A_{n1}=O_p\left(\frac{1}{n^3h^{3d}}\right)+O_p\left(\frac{1}{nh^{d-8}}\right)+O_p(h^8).
$$

Similarly, one can show that

$$
A_{n2}=O_p\left(\frac{1}{n^2h^{2d}}\right)+O_p\left(\frac{1}{nh^{d-4}}\right)+O_p(h^4).
$$

Hence

$$
T^*_{n12}=O_p\left(\frac{1}{n^2h^{2d}}\right)+O_p\left(\frac{1}{nh^{d-4}}\right)+O_p(h^4)=o_p(1).
$$

For $T^*_{n13}$, we have

$$
\begin{aligned}
T^*_{n13} &= \int_{\mathcal{C}}(\hat{f}_w^2(x)-f^2(x))^2v_0(x)dG(x)=\int_{\mathcal{C}}([\hat{f}_w(x)-f(x)+f(x)]^2-f^2(x))^2v_0(x)dG(x)\\
&\le 2\int_{\mathcal{C}}(\hat{f}_w(x)-f(x))^4v_0(x)dG(x)+8\int_{\mathcal{C}}(\hat{f}_w(x)-f(x))^2f^2(x)v_0(x)dG(x)\\
&= O_p\left(\frac{1}{nw^d}\right)+O_p(w^4).
\end{aligned}
$$

In summary, we get $T^*_{n11}=o_p(1)$, hence $T^*_{n1}=o_p(1)$. To show $T^*_{n2}=o_p(1)$, first we note that $T^*_{n2}$ is bounded above by

$$
\sup_{x\in\mathcal{C}}\frac{f^6(x)}{\hat{f}_w^6(x)}\cdot\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)(Y_i-m(X_i))(m(X_i)-\hat{m}(X_i))\hat{f}_h^2(X_i)\right\}^2 d\varphi(x).
$$

Note that the integral can be written as

$$
\begin{aligned}
&E\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\varepsilon_i\sqrt{v_0(X_i)}(m(X_i)\hat{f}_h(X_i)-\hat{m}(X_i)\hat{f}_h(X_i))\hat{f}_h(X_i)\right\}^2 d\varphi(x)\\
&=\frac{1}{n}\int E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)(m(X_1)\hat{f}_h(X_1)-\hat{m}(X_1)\hat{f}_h(X_1))^2\hat{f}_h^2(X_1)\right]d\varphi(x)\\
&\quad+\frac{(n-1)}{n}\int E\Big[K_{h1}(x)\varepsilon_1\sqrt{v_0(X_1)}(m(X_1)\hat{f}_h(X_1)-\hat{m}(X_1)\hat{f}_h(X_1))\hat{f}_h(X_1)\\
&\qquad\qquad K_{h2}(x)\varepsilon_2\sqrt{v_0(X_2)}(m(X_2)\hat{f}_h(X_2)-\hat{m}(X_2)\hat{f}_h(X_2))\hat{f}_h(X_2)\Big]d\varphi(x),
\end{aligned}
$$

and we have

$$
\begin{aligned}
&E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)(m(X_1)\hat{f}_h(X_1)-\hat{m}(X_1)\hat{f}_h(X_1))^2\hat{f}_h^2(X_1)\right]\\
&=E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{n}\sum_{j=1}^{n}K_{hj1}[m(X_1)-Y_j]\right)^2\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}\right]^2\right]\\
&\le 4E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{nh^d}K(0)[m(X_1)-Y_1]\right)^2\left[\frac{1}{nh^d}K(0)\right]^2\right]\\
&\quad+4E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{nh^d}K(0)[m(X_1)-Y_1]\right)^2\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}\right]^2\right]\\
&\quad+4E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{n}\sum_{j=2}^{n}K_{hj1}[m(X_1)-Y_j]\right)^2\left[\frac{1}{nh^d}K(0)\right]^2\right]\\
&\quad+4E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{n}\sum_{j=2}^{n}K_{hj1}[m(X_1)-Y_j]\right)^2\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}\right]^2\right].
\end{aligned}
$$

Note that

$$
E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{nh^d}K(0)[m(X_1)-Y_1]\right)^2\left[\frac{1}{nh^d}K(0)\right]^2\right]=O\left(\frac{1}{n^4h^{5d}}\right),
$$

$$
\begin{aligned}
&E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{nh^d}K(0)[m(X_1)-Y_1]\right)^2\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}\right]^2\right]\\
&=\frac{4K^2(0)}{n^4h^{2d}}\sum_{j=2}^{n}E\left[K_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)K_{hj1}^2\right]+\frac{4K^2(0)}{n^4h^{2d}}\sum_{j\neq k=2}^{n}E\left[K_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)K_{hj1}K_{hk1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{4(n-1)K^2(0)}{n^4h^{2d}}E\left[K_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)K_{h21}^2\right]+O\left(\frac{1}{n^2h^{2d}}\right)E\left[K_{h1}^2(x)\varepsilon_1^4 v_0^2(X_1)K_{h21}K_{h31}\right]\\
&=O\left(\frac{1}{n^3h^{4d}}\right)+O\left(\frac{1}{n^2h^{3d}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{n}\sum_{j=2}^{n}K_{hj1}[m(X_1)-Y_j]\right)^2\left[\frac{1}{nh^d}K(0)\right]^2\right]\\
&=\frac{K^2(0)}{n^2h^{2d}}E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}[m(X_1)-m(X_j)]\right]^2\right]\\
&\quad+\frac{K^2(0)}{n^2h^{2d}}E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}\sqrt{v(X_j)}\varepsilon_j\right]^2\right]\\
&=\frac{(n-1)K^2(0)}{n^4h^{2d}}E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)K_{h21}^2[m(X_1)-m(X_2)]^2\right]\\
&\quad+\frac{(n-1)(n-2)K^2(0)}{n^4h^{2d}}E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)K_{h21}K_{h21}[m(X_1)-m(X_2)][m(X_1)-m(X_3)]\right]\\
&\quad+\frac{(n-1)K^2(0)}{n^4h^{2d}}E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)K_{h21}^2 v_0(X_2)\varepsilon_2^2\right]\\
&=O\left(\frac{1}{n^2h^{3d-4}}\right)+O\left(\frac{1}{n^3h^{4d}}\right).
\end{aligned}
$$

Since $d<4$ as stated earlier in the assumptions, $nh^{d+4}\to\infty$ as $n\to\infty$, the above is of order $O\left(1/(n^2h^{3d-4})\right)$. Similarly, we can show that

$$
E\left[K_{h1}^2(x)\varepsilon_1^2 v_0(X_1)\left(\frac{1}{n}\sum_{j=2}^{n}K_{hj1}[m(X_1)-Y_j]\right)^2\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}\right]^2\right]=O\left(\frac{1}{h^{d-4}}\right).
$$

Note that

$$
\begin{aligned}
&E\Big[K_{h1}(x)\varepsilon_1\sqrt{v_0(X_1)}(m(X_1)\hat{f}_h(X_1)-\hat{m}(X_1)\hat{f}_h(X_1))\hat{f}_h(X_1)\cdot\\
&\qquad K_{h2}(x)\varepsilon_2\sqrt{v_0(X_2)}(m(X_2)\hat{f}_h(X_2)-\hat{m}(X_2)\hat{f}_h(X_2))\hat{f}_h(X_2)\Big]\\
&=E\Big[K_{h1}(x)\varepsilon_1\sqrt{v_0(X_1)}\left[\frac{1}{n}K_h(0)\varepsilon_1\sqrt{v_0(X_1)}+\frac{1}{n}K_{h21}[Y_2-m(X_1)]\right]\hat{f}_h(X_1)\\
&\quad K_{h2}(x)\varepsilon_2\sqrt{v_0(X_2)}\left[\frac{1}{nh^d}K(0)\varepsilon_2\sqrt{v_0(X_2)}+\frac{1}{n}K_{h21}[Y_1-m(X_2)]\right]\hat{f}_h(X_2)\Big]\\
&=\frac{K^2(0)}{n^2h^{2d}}E\left[K_{h1}(x)K_{h2}v_0(X_1)v_0(X_2)\hat{f}_h(X_1)\hat{f}_h(X_2)\right]\\
&\quad+\frac{1}{n^2}E\left[K_{h1,2}(x)K_{h12}^2v_0(X_1)v_0(X_2)\hat{f}_h(X_1)\hat{f}_h(X_2)\right]
\end{aligned}
$$

which has the order of $O\left(\frac{1}{n^2h^{2d}}\right)$. Therefore, we eventually get

$$
\sup_{x\in\mathcal{C}}\frac{f^6(x)}{\hat{f}_w^6(x)}\cdot\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)(Y_i-m(X_i))(m(X_i)-\hat{m}(X_i))\hat{f}_h^2(X_i)\right\}^2 d\varphi(x)
$$

to be the order of $O\left(n^{-1}h^{-d+4}\right)+O\left(n^{-2}h^{2-d}\right)$, and this implies $T_{n2}^{*}=o_p(1)$.

Finally, let's show that $T_{n3}^{*}=o_p(1)$. Note that

$$T_{n3}^{*}\le \sup_{x\in\mathcal{C}}\frac{f^6(x)}{\hat{f}_w^6(x)}\cdot\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[m(X_i)-\hat{m}(X_i)]^2\hat{f}_h^2(X_i)\right\}^2 d\varphi(x).$$

We also have

$$\begin{aligned}
&E\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[m(X_i)-\hat{m}(X_i)]^2\hat{f}_h^2(X_i)\right\}^2 d\varphi(x)\\
&=\frac{1}{n}\int_{\mathcal{C}}E\left\{K_{h1}^2(x)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}[m(X_1)-Y_j]\right]^4\right\}d\varphi(x)\\
&\quad+\frac{n-1}{n}\int_{\mathcal{C}}E\left\{K_{h1,2}(x)\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl1}[m(X_1)-Y_l]\right]^2\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl2}[m(X_2)-Y_l]\right]^2\right\}d\varphi(x).
\end{aligned}$$

By C-R inequality, we have

$$\begin{aligned}
&\frac{1}{n}\int_{\mathcal{C}}E\left\{K_{h1}^2(x)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}[m(X_1)-Y_j]\right]^4\right\}d\varphi(x)\\
&\le\frac{8}{n}\int_{\mathcal{C}}E\left\{K_{h1}^2(x)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}\varepsilon_j\sqrt{v(X_j)}\right]^4\right\}d\varphi(x)\\
&\quad+\frac{8}{n}\int_{\mathcal{C}}E\left\{K_{h1}^2(x)\left[\frac{1}{n}\sum_{j=2}^{n}K_{hj1}[m(X_1)-m(X_j)]\right]^4\right\}d\varphi(x)=O\left(\frac{1}{n^5h^{5d}}\right)+O\left(\frac{1}{nh^{d-8}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{C}}E\left\{K_{h1,2}(x)\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl1}[m(X_1)-Y_l]\right]^2\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl2}[m(X_2)-Y_l]\right]^2\right\}d\varphi(x)\\
&\le 4\int_{\mathcal{C}}E\left\{K_{h1,2}(x)\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl1}[m(X_1)-m(X_l)]\right]^2\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl2}[m(X_2)-m(X_l)]\right]^2\right\}d\varphi(x)\\
&\quad+4\int_{\mathcal{C}}E\left\{K_{h1,2}(x)\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl1}\varepsilon_l\sqrt{v(X_l)}\right]^2\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl2}[m(X_2)-m(X_l)]\right]^2\right\}d\varphi(x)\\
&\quad+4\int_{\mathcal{C}}E\left\{K_{h1,2}(x)\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl1}[m(X_1)-m(X_l)]\right]^2\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl2}\varepsilon_l\sqrt{v(X_l)}\right]^2\right\}d\varphi(x)\\
&\quad+4\int_{\mathcal{C}}E\left\{K_{h1,2}(x)\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl1}\varepsilon_l\sqrt{v(X_l)}\right]^2\left[\frac{1}{n}\sum_{l=1}^{n}K_{hl2}\varepsilon_l\sqrt{v(X_l)}\right]^2\right\}d\varphi(x)
\end{aligned}$$

$$= O(h^8) + O\left(\frac{1}{n^2h^{2d-4}}\right) + O\left(\frac{1}{n^4h^{4d}}\right).$$

Hence

$$T_{n3}^* = O_p\left(\frac{1}{n^5h^{5d}}\right) + O_p\left(\frac{1}{nh^{d-8}}\right) + O_p(h^8) + O_p\left(\frac{1}{n^2h^{2d-4}}\right) + O_p\left(\frac{1}{n^4h^{4d}}\right) = o_p(1).$$

□

*The proof of Theorem 3.2.* Similar to Koul and Ni (2004), it suffices to show

$$\sup_\theta |T_n(\theta) - T_n^*(\theta)| = o_p(1).$$

Adding and subtracting $v(X_i,\theta)$ from $(Y_i - \hat{m}(X_i))^2$ inside the integrand of $T_n^*(\theta)$, $T_n^*(\theta)$ can be written as the sum of $T_n(\theta)$, $C_n(\theta)$ which is defined by

$$C_n(\theta) = \int_{\mathcal{C}} \left[\frac{\sum_{i=1}^n K_{hi}(x)v(X_i,\theta)\hat{f}_h^2(X_i)}{n\hat{f}_w^3(x)} - v(x,\theta)\right]^2 dG(x),$$

and another term which is bounded above in absolute value by $2(C_n(\theta)T_n(\theta))^{1/2}$. Therefore,

$$\sup_\theta |T_n^*(\theta) - T_n(\theta)| \le \sup_\theta C_n(\theta) + 2\sup_\theta (C_n(\theta)T_n(\theta))^{1/2}$$

It thus suffices to show that $\sup_\theta C_n(\theta) = o_p(1)$, and $\sup_\theta T_n(\theta) = O_p(1)$. Note that

$$\begin{aligned}
C_n(\theta) &= \int_{\mathcal{C}} \left[\frac{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)v(X_i,\theta)\hat{f}_h^2(X_i)}{\hat{f}_w^3(x)} - v(x,\theta)\right]^2 dG(x) \\
&= \int_{\mathcal{C}} \left[\frac{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)v(X_i,\theta)\hat{f}_h^2(X_i)}{\hat{f}_w(x)} - (\hat{f}_w^2(x) - f^2(x) + f^2(x))v(x,\theta)\right]^2 \hat{f}_w^{-4}(x)dG(x) \\
&\le 4\int_{\mathcal{C}} \left[\frac{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)v(X_i,\theta)f^2(X_i)}{\hat{f}_w(x)} - f^2(x)v(x,\theta)\right]^2 \hat{f}_w^{-4}(x)dG(x) \\
&\quad +4\int_{\mathcal{C}} \left[\frac{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)v(X_i,\theta)(\hat{f}_h^2(X_i) - f^2(X_i))}{\hat{f}_w(x)}\right]^2 \hat{f}_w^{-4}(x)dG(x) \\
&\quad +2\int_{\mathcal{C}} [(\hat{f}_w^2(x) - f^2(x))v(x,\theta)]^2 \hat{f}_w^{-4}(x)dG(x) \\
&:= 4C_{n1}(\theta) + 4C_{n2}(\theta) + 2C_{n3}(\theta)
\end{aligned}$$

For $C_{n1}(\theta)$, we have

$$
\begin{aligned}
C_{n1}(\theta) &= \int_{\mathcal{C}} \left[ \frac{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) v(X_i,\theta) f^2(X_i)}{\hat{f}_w(x)} - f^2(x) v(x,\theta) \right]^2 \hat{f}_w^{-4}(x) dG(x) \\
&\le \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) v(X_i,\theta) f^2(X_i) - \hat{f}_w(x) f^2(x) v(x,\theta) \right]^2 \hat{f}_w^{-6}(x) dG(x) \\
&\le \sup_{x\in\mathcal{C}} \frac{f^6(x)}{\hat{f}_w^6(x)} \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) v(X_i,\theta) f^2(X_i) - \hat{f}_w(x) f^2(x) v(x,\theta) \right]^2 d\varphi(x).
\end{aligned}
$$

Treating $f^2(x)v(x,\theta)$ as $m(x,\theta)$ and using the same arguments as in Koul and Ni (2004), one can show that $\sup_\theta C_{n1}(\theta) = o_p(1)$.

To show that $\sup_\theta C_{n2}(\theta) = o_p(1)$, first we note that for any fixed $\theta \in \Theta$, $C_{n2}(\theta) = o_p(1)$ which can be proved in the same manner as showing $T^*_{n12} = o_p(1)$ in the proof of Theorem 3.1, so it suffices to show that $C_{n2}(\theta)$ is uniformly continuous in $\theta \in \Theta$. Adding and subtracting $v(X_i,\theta_2)$, we have

$$
\begin{aligned}
C_{n2}(\theta_1) &= \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) v(X_i,\theta_1)(\hat{f}_h^2(X_i) - f^2(X_i)) \right]^2 \hat{f}_w^{-6}(x) dG(x) \\
&= \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)(v(X_i,\theta_1) - v(X_i,\theta_2) + v(X_i,\theta_2))(\hat{f}_h^2(X_i) - f^2(X_i)) \right]^2 d\hat{\varphi}_w(x).
\end{aligned}
$$

Thus $|C_{n2}(\theta_1) - C_{n2}(\theta_2)|$ is bounded above by

$$
\begin{aligned}
&\sup_{x\in\mathcal{C}} \left| \frac{f^6(x)}{\hat{f}_w^6(x)} \right| \left| \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)(v(X_i,\theta_1) - v(X_i,\theta_2))(\hat{f}_h^2(X_i) - f^2(X_i)) \right]^2 d\varphi(x) \right. \\
&+2\int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)(v(X_i,\theta_1) - v(X_i,\theta_2))(\hat{f}_h^2(X_i) - f^2(X_i)) \right] \\
&\left. \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) v(X_i,\theta_2)(\hat{f}_h^2(X_i) - f^2(X_i)) \right] d\varphi(x) \right|
\end{aligned}
$$

By assumption (v3), $|C_{n2}(\theta_1) - C_{n2}(\theta_2)|$ is further bounded above by

$$
\begin{aligned}
&\|\theta_2 - \theta_1\|^2 \sup_{x\in\mathcal{C}} \left| \frac{f^6(x)}{\hat{f}_w^6(x)} \right| \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| l(X_i) \right]^2 d\varphi(x) \\
&+\|\theta_2 - \theta_1\| \sup_{x\in\mathcal{C}} \left| \frac{f^6(x)}{\hat{f}_w^6(x)} \right| \int_{\mathcal{C}} \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| l(X_i) \right] \\
&\qquad \cdot \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| v(X_i,\theta_2) \right] d\varphi(x) \qquad (2.1)
\end{aligned}
$$

Note that for a fixed $\theta_3 \in \Theta$ which is compact, there exist a non-negative function $L$ such that

$$|v(\theta_2, X_i) - v(\theta_3, X_i) + v(\theta_3, X_i)| \le \|\theta_2 - \theta_3\| l(X_i) + v(\theta_3, X_i) \le L(X_i)$$

With similar argument in $T_{n2}^*$, we know the integration on the right hand side (RHS) of (2.1) is the order of $o_p(1)$. Thus it suffices to show that

$$\int_{\mathcal{C}} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| l(X_i) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| L(X_i) \right] d\varphi(x)$$

has the order of $o_p(1)$. Taking expectation, we have

$$\begin{aligned} & E \int_{\mathcal{C}} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| l(X_i) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) |\hat{f}_h^2(X_i) - f^2(X_i)| L(X_i) \right] d\varphi(x) \\ = & E \int_{\mathcal{C}} \left[ \frac{1}{n^2} \sum_{i=1}^{n} K_{hi}^2(x) |\hat{f}_h^2(X_i) - f^2(X_i)|^2 l(X_i) L(X_i) \right] \\ & + \left[ \frac{2}{n^2} \sum_{i \le j} K_{hi,j}(x) L(X_i) l(X_j) |\hat{f}_h^2(X_i) - f^2(X_i)| |\hat{f}_h^2(X_j) - f^2(X_j)| \right] d\varphi(x) := C_1 + C_2 \end{aligned}$$

Note that

$$C_1 = \frac{1}{n} \int_{\mathcal{C}} E \left[ K_{h1}^2(x) |\hat{f}_h^2(X_1) - f^2(X_1)|^2 l(X_1) L(X_1) \right] d\varphi(x).$$

Recall $A_{n1}$, with exact same technique, adding and subtracting $f(x)$ inside $\hat{f}_h^2(x)$, and then adding and subtracting $E(K_h(X_2 - X_1)|X_1)$ in the sum, while functions $K,\ l,\ L,\ f$ are continuous and bounded, we get $C_1 = o(1)$

Next by Cauchy-Schwartz Inequality, $C_2$ is bounded above by

$$c \int_{\mathcal{C}} \sqrt{E K_{h1}^2(x) L^2(X_1) (\hat{f}_h^2(X_1) - f^2(X_1))^2} \sqrt{E K_{h2}^2(x) L^2(X_2) (\hat{f}_h^2(X_2) - f^2(X_2))^2} d\varphi(x)$$

for a constant $c$. Again, with similar technique applied to $A_{n1}$, and $nh^{2d} \to \infty$, $C_2 = o(1)$. Together we know that the second part is $o_p(1)$, and this proves uniform continuity of $C_{n2}(\theta)$.

Since $v(x, \theta)$ is bounded on the compact set $\mathcal{C} \times \Theta$, so $\sup_{\theta \in \Theta} C_{n3}(\theta) = o_p(1)$ is straightforward.

Next, adding and subtracting $v_0(X_i)$ in $T_n(\theta)$, we obtain

$$\begin{aligned} T_n(\theta) \le 2 \sup_{x \in \mathcal{C}} \frac{f^6(x)}{\hat{f}_w^6(x)} \Bigg\{ & \int_{\mathcal{C}} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) [v_0(X_i) - v(X_i, \theta)] \hat{f}_h^2(X_i) \right]^2 d\varphi(x) \\ & + \int_{\mathcal{C}} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) [(Y_i - \hat{m}(X_i))^2 - v_0(X_i)] \hat{f}_h^2(X_i) \right]^2 d\varphi(x) \Bigg\} \end{aligned}$$

Note that the second term on the RHS of the above expression is indeed $T_n(\theta_0)$ which is $o_p(1)$. Finally we can show that for a positive constant $c$,

$$\int_{\mathcal{C}} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)[v_0(X_i) - v(X_i,\theta)]\hat{f}_h^2(X_i) \right]^2 d\varphi(x)$$
$$\leq c \int_{\mathcal{C}} \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) l(X_i)\hat{f}_h^2(X_i) \right]^2 d\varphi(x)$$

which is $O_p(1)$ by the compactness of $\Theta$. This concludes proof of $\sup_{\theta\in\Theta} T_n(\theta) = O_p(1)$, hence the proof of Theorem 3.2. □

To show the asymptotic normality of $\hat{\theta}_n$, we introduce the following notations, for the sake of convenience, and they maybe used in the context that follows.

$$\begin{aligned}
&\dot{\mu}_n(x,\theta) := \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta) f^2(X_i),\\
&\dot{\mu}_n(x,\theta_0) := \dot{\mu}_n(x),\\
&\dot{\mu}_h(x) := E\dot{\mu}_n(x,\theta_0) = EK_h(x-X)\dot{v}(\theta_0,X) f^2(X)\\
&\Sigma_n(b) = \int [b^T \dot{\mu}_n(x,\theta_0)]^2 d\varphi(x)\\
&U_n(x,\theta) = \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[(Y_i - \hat{m}(X_i))^2 - v(X_i,\theta)]^2 \hat{f}_h^2(X_i),\\
&U_n(x) := U_n(x,\theta_0),\\
&D_n(\theta) := \int Z_n^2(x,\theta) d\hat{\varphi}_w(x) := \int \left[\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)(v(X_i,\theta_0) - v(X_i,\theta))\hat{f}_h^2(X_i)\right]^2 d\hat{\varphi}_w(x),\\
&\xi(X_i) := (Y_i - m(X_i))^2 - v_0(X_i).
\end{aligned}$$

We will use $\Delta_{ig} := \hat{g}(X_i) - g(X_i)$ to denote any difference between the function at sample point $X_i$ and its approximation at point $X_i$. The approximation methods are either kernel smoothing estimates for densities and N-W estimates for regression functions.

*The proof of Theorem 3.3.* First show $nh^d\|\hat{\theta}_n - \theta_0\|^2 = O_P(1)$. To proceed, we could show that $nh^d D_n(\hat{\theta}_n) = O_p(1)$. Note that by adding and subtracting $(Y_i - \hat{m}(X_i))^2$, we can see that $D_n(\hat{\theta}_n)$ is bounded above by $2[T_n(\theta_0) + T_n(\hat{\theta}_n)]$, where

$$T_n(\theta) = \int \left[ \frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)((Y_i - \hat{m}(X_i))^2 - v(X_i,\theta))\hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x).$$

It suffices to show that $nh^d T_n(\theta_0) = O_p(1)$, together by definition, $\hat{\theta}_n = \text{argmin}_{\theta\in\Theta} T_n(\theta)$, we know that $nh^d T_n(\hat{\theta}_n) = O_p(1)$, and therefore $nh^d\|\hat{\theta}_n - \theta_0\|^2 = O_P(1)$. Adding and subtracting $m(X_i)$

$$
\begin{aligned}
nh^d T_n(\theta_0) &= nh^d \int \left[ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)((Y_i - \hat{m}(X_i))^2 - v_0(X_i)) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&\leq 3nh^d \int \left[ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)((Y_i - m(X_i))^2 - v_0(X_i)) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&\quad + 6nh^d \int \left[ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)((Y_i - m(X_i))(m(X_i) - \hat{m}(X_i)) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&\quad + 3nh^d \int \left[ \frac{1}{n} \sum_{i=1}^n K_{hi}(x)(m(X_i) - \hat{m}(X_i))^2) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&:= 3nh^d T_{n1}(\theta_0) + 6nh^d T_{n2}(\theta_0) + 3nh^d T_{n3}(\theta_0).
\end{aligned}
$$

Note that:

$$
\begin{aligned}
T_{n1}(\theta_0) &= \int \left[ \frac{1}{n^2} \sum_{i=1}^n K_{hi}(x)(\varepsilon_i^2 - 1) v_0(X_i) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&= \int \frac{1}{n^2} \sum_{i=1}^n K_{hi}^2(x)(\varepsilon_i^2 - 1)^2 v_0^2(X_i) \hat{f}_h^4(X_i) d\hat{\varphi}_w(x) \\
&\quad + \int \frac{1}{n^2} \sum_{i \neq j}^n K_{hi}(x) K_{hj}(x)(\varepsilon_i^2 - 1)(\varepsilon_j^2 - 1) v_0(X_i) v_0(X_j) \hat{f}_h^2(X_i) \hat{f}_h^2(X_j) d\hat{\varphi}_w(x).
\end{aligned}
$$

$$
\begin{aligned}
ET_{n1}(\theta_0) &= \int \frac{1}{n} E[K_{h1}^2(x) E((\varepsilon_1^2 - 1)^2 | X_1) v_0^2(X_1) \hat{f}_h^4(X_1)] d\hat{\varphi}_w(x) \\
&+ \int \frac{n(n-1)}{n^2} E[K_{h1,2}(x) E((\varepsilon_1^2 - 1)|X_1) E((\varepsilon_2^2 - 1)|X_2) v_0(X_1) v_0(X_2) \hat{f}_h^2(X_1) \hat{f}_h^2(X_2)] d\hat{\varphi}_w(x)
\end{aligned}
$$

which is the order of $O\left(\frac{1}{nh^d}\right)$. The second term equals to zero under null assumption and the independence between any paired $X_i's$, therefore we have $nh^d T_{n1}(\theta_0) = O_p(1)$. Also it is easy to see that $T_{n2}(\theta_0) = T_{n2}^*(\theta_0)$, and $T_{n3}(\theta_0) = T_{n3}^*(\theta_0)$. Utilizing the previous result, with the restriction that $d < 4$, we could see that $nh^d T_n(\theta_0) = O_p(1)$, thus leading to $nh^d D_n(\hat{\theta}_n) = O_p(1)$.

Next we shall show that $\forall\ 0 < a < \infty$, $\exists N_a$, such that

$$
P\left( \frac{D_n(\hat{\theta}_n)}{\|\hat{\theta}_n - \theta_0\|^2} \geq a + \inf_{\|b\|=1} b^T \Sigma_0 b \right) > 1 - a \tag{2.2}
$$

The claim will follow with the $nh^d D_n(\hat{\theta}_n) = O_p(1)$, and the fact that $\Sigma_0$ is positive definite. Let $u_n := (\hat{\theta}_n - \theta_0), d_{ni} := v(\hat{\theta}_n, X_i) - v(X_i, \theta_0) - u_n^T \dot{v}(X_i, \theta_0), i = 1, 2, \ldots, n$. Then $D_n(\hat{\theta}_n)/\|\hat{\theta}_n - \theta_0\|^2$

can be written as

$$
\begin{aligned}
&\int \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \left( \frac{\dot{v}(X_i, \theta_0) - \dot{v}(\hat{\theta}_n, X_i) + u_n^T \dot{v}(X_i, \theta_0) - u_n^T \dot{v}(X_i, \theta_0)}{\|\hat{\theta}_n - \theta_0\|} \right) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&= \int \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \left( \frac{\dot{v}(X_i, \theta_0) - \dot{v}(\hat{\theta}_n, X_i) - u_n^T \dot{v}(X_i, \theta_0)}{\|\hat{\theta}_n - \theta_0\|} \right) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&\quad + \int \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \left( \frac{u_n^T \dot{v}(X_i, \theta_0)}{\|\hat{\theta}_n - \theta_0\|} \right) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&\quad + 2 \int \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \left( \frac{d_{ni}}{\|u_n\|} \right) \hat{f}_h^2(X_i) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \left( \frac{u_n^T \dot{v}(X_i, \theta_0)}{\|u_n\|} \right) \hat{f}_h^2(X_i) \right] d\hat{\varphi}_w(x) \\
&:= \int \bar{D}_{n1}^2 d\hat{\varphi}_w(x) + \int \bar{D}_{n2}^2 d\hat{\varphi}_w(x) + 2 \int \bar{D}_{n1} \bar{D}_{n2} d\hat{\varphi}_w(x) \\
&\geq \int \bar{D}_{n1}^2 d\hat{\varphi}_w(x) + \int \bar{D}_{n2}^2 d\hat{\varphi}_w(x) - 2 \sqrt{\int \bar{D}_{n1}^2 d\hat{\varphi}_w(x) \int \bar{D}_{n2}^2 d\hat{\varphi}_w(x)} \\
&:= D_{n1} + D_{n2} - 2\sqrt{D_{n1} D_{n2}}.
\end{aligned}
$$

From the above derivation, it suffices to show that $D_{n2}$ is bounded above, bounded below by $\inf_{\|b\|=1} \Sigma_n(b)$, and $D_{n1} = o_p(1)$. By Cauchy-Schwartz inequality,

$$
D_{n2} = \int \left[ \frac{u_n^T \dot{\mu}_n(x, \theta_0)}{\|u_n\|} \right]^2 d\hat{\varphi}_w(x) \leq \int \dot{\mu}_n^T(x, \theta_0) \dot{\mu}_n(x, \theta_0) d\hat{\varphi}_w(x).
$$

With $K$, $\dot{v}$, $f$ bounded above, we know that $D_{n2}$ is bounded above. By assumption (v4), and the fact that $\hat{\theta}_n$ is a consistent estimator of $\theta$, we know $D_{n1} = o_p(1)$.

Next we show $D_{n2} \geq \inf_{\|b\|=1} \Sigma_n(b)$, and $\Sigma_n(b) \to b^T \Sigma b$, for any $q$ dimension vector $b$.

$$
\begin{aligned}
&\int \left[ b^T \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) \hat{f}_h^2(X_i) \right]^2 d\hat{\varphi}_w(x) \\
&= b^T \int \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) \hat{f}_h^2(X_i) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) \hat{f}_h^2(X_i) \right]^T d\hat{\varphi}_w(x) b.
\end{aligned}
$$

Plus and minus $f^2(X_i)$ after $\hat{f}_h^2(X_i)$ gives the following,

$$
\begin{aligned}
&\left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) \hat{f}_h^2(X_i) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) \hat{f}_h^2(X_i) \right]^T \\
&= \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) (\hat{f}_h^2(X_i) - f^2(X_i)) \right] \left[ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x) \dot{v}(X_i, \theta_0) (\hat{f}_h^2(X_i) - f^2(X_i)) \right]^T
\end{aligned}
$$

$$+2\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta_0)(\hat{f}_h^2(X_i)-f^2(X_i))\right]\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta_0)f^2(X_i)\right]^T$$
$$+\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta_0)f^2(X_i)\right]\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta_0)f^2(X_i)\right]^T.$$

As proved in $A_{n1}$ and $A_{n2}$, with $\dot{v}$ continuous and bounded, we know the first two term on the RHS goes to zero in probability. Also note that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta_0)f^2(X_i)=f^3(x)\dot{v}(x,\theta)$$

and the fact that $\sup_x|f^6(x)\hat{f}_w^{-6}(x)-1|=o_p(1)$, this concludes $\Sigma_n(b)\to b^T\Sigma b$.

Now we will show that $\Sigma_n(b)$ is uniform continuous in $b$. For any $\delta>0$ and any two unit vectors $b_1,b_2$, such that $\|b_1-b_2\|\le\delta$, we see that $|\Sigma_n(b_1)-\Sigma_n(b_2)|$ is bounded above by

$$\begin{aligned}
&\sup_{x\in\mathcal{C}}\left|\frac{f^6(x)}{\hat{f}_w^6(x)}\right|\left|\int_{\mathcal{C}}[b_1^T\dot{\mu}_n(x,\theta_0)]^2d\varphi(x)-\int_{\mathcal{C}}[b_2^T\dot{\mu}_n(x,\theta_0)]^2d\varphi(x)\right|\\
&=\sup_{x\in\mathcal{C}}\left|\frac{f^6(x)}{\hat{f}_w^6(x)}\right|\left|\int_{\mathcal{C}}[(b_1-b_2)^T\dot{\mu}_n(x,\theta_0)]^2d\varphi(x)+2\int_{\mathcal{C}}[(b_1-b_2)^T\dot{\mu}_n(x,\theta_0)][b_2^T\dot{\mu}_n(x,\theta_0)]d\varphi(x)\right|\\
&\le\delta(\delta+2)\sup_{x\in\mathcal{C}}\left|\frac{f^6(x)}{\hat{f}_w^6(x)}\right|\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta_0)\hat{f}_h^2(X_i)\right]^2d\varphi(x).
\end{aligned}$$

Again, add and subtract $f^2(X_i)$ after $\hat{f}_h^2(X_i)$, similarly we can show that $\sup_{\|b\|=1}|\Sigma_n(b_1)-b^T\Sigma_0 b|=o_p(1)$. This, together with the fact that $D_{n2}\ge\inf_{\|b\|=1}\Sigma_n(b)$, implies that $nh^d\|\hat{\theta}_n-\theta_0\|^2=O_p(1)$ by routine arguments.

Note that $\hat{\theta}_n$ satisfies the equation $\partial T_n(\theta)/\partial\theta|_{\theta=\hat{\theta}_n}=0$. Adding and subtracting $v_0(X_i)$ inside $U_n(x,\theta)$, we obtain

$$\int_{\mathcal{C}}U_n(x)\dot{\mu}_n(x,\hat{\theta}_n)d\hat{\varphi}_w(x)=\int_{\mathcal{C}}Z_n(x,\hat{\theta}_n)\dot{\mu}_n(x,\hat{\theta}_n)d\hat{\varphi}_w(x)\tag{2.3}$$

We may discover that the left hand side (LHS) of (2.3) is equivalent to

$$\begin{aligned}
&\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[(Y_i-m(X_i)+m(X_i)-\hat{m}(X_i))^2-v_0(X_i)][\hat{f}_h^2(X_i)-f^2(X_i)+f^2(X_i)]\right]\\
&\qquad\cdot\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta)[\hat{f}_h^2(X_i)-f^2(X_i)+f^2(X_i)]\right]d\hat{\varphi}_w(x)\\
&=\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\xi(X_i)+(m(X_i)-\hat{m}(X_i))^2+2\varepsilon_i\sqrt{v_0(X_i)}(m(X_i)-\hat{m}(X_i))]\right.
\end{aligned}$$

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$$\cdot[\hat{f}_h^2(X_i) - f^2(X_i) + f^2(X_i)]\Bigg]\left[\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)[\hat{f}_h(X_i) - f(X_i) + f(X_i)]^2\right]d\hat{\varphi}_w(x)$$

$$:= \int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i) + \Delta_{im}^2 + 2\varepsilon_i\sqrt{v(X_i,\theta)}\Delta_{im}][\hat{f}_h^2(X_i) - f^2(X_i) + f^2(X_i)]\right]$$

$$\cdot\left[\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}_0(X_i)[\Delta_{if}^2 + f^2(X_i) + 2\Delta_{if}f(X_i)]\right]d\hat{\varphi}_w(x)$$

$$= \int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)f^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f^2(X_i)\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)f(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f^2(X_i)\right\}d\hat{\varphi}_w(x)$$

$$+\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)\Delta_{if}^2]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f^2(X_i)\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\varepsilon_i\sqrt{v_0(X_i)}\Delta_{im}\hat{f}_h(X_i)f(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f^2(X_i)\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\varepsilon_i\sqrt{v_0(X_i)}\Delta_{im}\hat{f}_h(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f^2(X_i)\right\}d\hat{\varphi}_w(x)$$

$$+\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\Delta_{im}^2\hat{f}_h^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f^2(X_i)\right\}d\hat{\varphi}_w(x)$$

$$+\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)f^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)f(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x)$$

$$+\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)\Delta_{if}^2]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\varepsilon_i\sqrt{v_0(X_i)}\Delta_{im}\hat{f}_h(X_i)f(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\varepsilon_i\sqrt{v_0(X_i)}\Delta_{im}\hat{f}_h(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x)$$

$$+\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\Delta_{im}^2\hat{f}_h^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x)$$

$$+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[\xi(X_i)f^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\hat{\varphi}_w(x)$$

$$
\begin{aligned}
&+4\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\xi(X_i)f(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\xi(X_i)\Delta_{if}^2]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\hat{\varphi}_w(x)\\
&+4\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\varepsilon_i\sqrt{v_0(X_i)}\Delta_{im}\hat{f}_h(X_i)f(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\hat{\varphi}_w(x)\\
&+4\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\varepsilon_i\sqrt{v_0(X_i)}\Delta_{im}\hat{f}_h(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\Delta_{im}^2\hat{f}_h^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\hat{\varphi}_w(x).
\end{aligned}
$$

For the sake of convenience, we will denote these 18 terms as $S_{n1},\cdots,S_{n18}$.

Next, we would, on each term $S_{n1}$ to $S_{n6}$, add and subtract $\dot{\mu}_n(x,\theta)$, $\dot{\mu}_n(x,\theta_0)$ after $\dot{\mu}_h(x,\theta_0)$, add and subtract $f^{-6}(x)$ to $\hat{\varphi}_w(x)$. The main structure after such derivation are denoted with $\tilde{S}$. For the rest, we could show that after multiplying by $\sqrt{n}$, they converges to zero fast enough so that no component is left for variance. First note that,

$$
\begin{aligned}
n\tilde{S}_{n2}^2 &= n\left(\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\xi(X_i)f(X_i)\Delta_{if}\right]\dot{\mu}_h(x,\theta_0)d\varphi(x)\right)^2\\
&= n\int_{\mathcal{C}}\left[\frac{1}{n^2}\sum_{i=1}^{n}K_{hi}(x)K_{hi}(y)\xi^2(X_i)f^2(X_i)(\hat{f}_h(X_i)-f(X_i))^2\right]\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(y,\theta_0)d\varphi(x)d\varphi(y)\\
&\quad +n\int_{\mathcal{C}}\left[\frac{1}{n^2}\sum_{i\neq j}^{n}K_{hi}(x)K_{hi}(y)\xi(X_i)\xi(X_j)f(X_i)f(X_j)(\hat{f}_h(X_i)-f(X_i))(\hat{f}_h(X_j)-f(X_j))\right]\\
&\qquad\qquad \cdot\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(y,\theta_0)d\varphi(x)d\varphi(y).
\end{aligned}
$$

Note that by taking conditional expectation on $X_i$, and $X_j$ respectively, the cross product term equals to zero under the Null hypothesis. Note that every element inside the integral with random variable $X_i's$ are positive, by Fubini's theorem, adding and subtracting $E[K_{hi1}|X_1]$, we see that $En\tilde{S}_{n2}^2$ equals

$$
\begin{aligned}
&n\int_{\mathcal{C}}E\left[\frac{1}{n^2}\sum_{i=1}^{n}K_{hi}(x)K_{hi}(y)\xi^2(X_i)f^2(X_i)(\hat{f}_h(X_i)-f(X_i))^2\right]\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(y,\theta_0)d\varphi(x)d\varphi(y)\\
&=\int_{\mathcal{C}}E\Bigg[K_{h1}(x)K_{h1}(y)\xi_1^2f^2(X_1)\left(\frac{1}{n}\sum_{i=2}^{n}K_{hi1}-E(K_{hi1}|X_1)\right)^2\\
&\quad +2K_{h1}(x)K_{h1}(y)\xi^2(X_1)f^2(X_1)\left(\frac{1}{n}\sum_{i=2}^{n}K_{hi1}-E(K_{hi1}|X_1)\right)\Big(E(K_{hi1}|X_1)-f(X_1)\Big)
\end{aligned}
$$

$$
+K_{h1}(x)K_{h1}(y)\xi^2(X_1)f^2(X_1)\bigg(E(K_{hi1}|X_1)-f(X_1)\bigg)^2\Bigg]\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(y,\theta_0)d\varphi(x)d\varphi(y)+O\Big(\frac{1}{nh^d}\Big)
$$
$$
=O\Big(\frac{1}{nh^{2d}}\Big)+O\Big(\frac{1}{n^2h^{3d}}\Big)+O(h^4)
$$

where the last equality comes from $\dot{\mu}_h(x,\theta_0)$ and $\dot{\mu}_h(y,\theta_0)$ is finite on $\mathcal{C}$. For $S_{n3}^2, S_{n4}^2, S_{n5}^2$, we may observe the similar pattern where the expectation of cross product term equals zero when conditioned on $X_i, X_j$. So, for simplicity, we will show the form of $nS_{n3}^2, nS_{n4}^2, nS_{n5}^2$ where they all converges to zero.

Next, for $nS_{n3}^2$, add and subtract $E(K_h(X_j-X_i)|X_1)$ in $(\hat{f}_{hij})^4$ of the $En\tilde{S}_{n3}^2$ will gives us the following derivation.

Since,

$$
n\tilde{S}_{n3}^2 = n\int_{\mathcal{C}}\Big\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\xi(X_i)(\hat{f}_h(X_i)-f(X_i))^2\Big\}\Big\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(y)\xi(X_i)(\hat{f}_h(X_i)-f(X_i))^2\Big\}
$$
$$
\cdot\dot{\mu}_h(x,\theta)\dot{\mu}_h(y,\theta)d\varphi(x)d\varphi(y)
$$

Therefore, we may calculate the expectation,

$$
\begin{aligned}
En\tilde{S}_{n3}^2 &= \int_{\mathcal{C}} EK_{h1}(x)K_{h1}(y)\xi^2(X_1)(\hat{f}_h(X_i)-f(X_1))^4\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\varphi(x)d\varphi(y)\\
&\le 8\int_{\mathcal{C}} EK_{h1}(x)K_{h1}(y)\xi^2(X_1)(\frac{1}{n}\sum_{j=1}^{n}K_{hj1}-E(K_{hj1}|X_1)^4\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\varphi(x)d\varphi(y)\\
&\quad +8\int_{\mathcal{C}} EK_{h1}(x)K_{h1}(y)\xi^2(X_1)(E(K_{hj1}|X_1)-f(X_1))^4\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\varphi(x)d\varphi(y).
\end{aligned}
$$

Then similar to $A_{n1}$, we know that $nS_{n3}^2 = O_p((n^3h^{3d})^{-1}) + O_p((nh^{d-8})^{-1})$.

$$
En\tilde{S}_{n4}^2 = \int_{\mathcal{C}} E\Bigg[K_{hi}(x)K_{hi}(y)\varepsilon_1^2 v_0(X_1)(\frac{1}{n}\sum_{j=1}^{n}K_{hj1}(Y_j-m(X_1)))^2f^2(X_i)\Bigg]
$$
$$
\cdot\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\varphi(x)d\varphi(y)
$$

And by technique similar to that of $T_{n2}^*$, we may show that the order is $O(h^4)+O(1/(n^2h^{2d-4}))$.

$$
\begin{aligned}
En\tilde{S}_{n5}^2 &= \int_{\mathcal{C}} EK_{hi}(x)K_{hi}(y)\varepsilon_i^2 v_0(X_i)\Big(\frac{1}{n}\sum_{i=1}^{n}K_{hj1}(m(X_j)-m(X_1)+\varepsilon_j\sqrt{v_0(X_j)})\Big)^2\\
&\qquad (\hat{f}_h(X_i)-f(X_i))^2\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\varphi(x)d\varphi(y)\\
&= O(h^8)+O\Big(\frac{1}{nh^{d-4}}\Big)+O\Big(\frac{1}{n^2n^{2d}}\Big)
\end{aligned}
$$

Also note that $En\tilde{S}_{n6}^2$ equals

$$
\begin{aligned}
& n\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[\Delta_{im}^2\hat{f}_h^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(y)[\Delta_{im}^2\hat{f}_h^2(X_i)]\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&= n\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hij}(Y_j-m(X_i))\right]^2\right\}\\
&\qquad\left\{\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hij}(Y_j-m(X_i))\right]^2\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&= n\int_{\mathcal{C}}\left\{\frac{1}{n^2}\sum_{i=1}^{n}K_{hi}(x)K_{hi}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hij}(Y_j-m(X_i))\right]^4\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&\quad +n\int_{\mathcal{C}}\left\{\frac{1}{n^2}\sum_{i\neq k}^{n}K_{hi}(x)K_{hj}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hij}(Y_j-m(X_i))\right]^2\right.\\
&\qquad\qquad\left.\cdot\left[\frac{1}{n}\sum_{k=1}^{n}K_{hki}(Y_k-m(X_i))\right]^2\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&:= nS_{n61}+nS_{n62}
\end{aligned}
$$

First we have

$$
\begin{aligned}
EnS_{n61} &= \int_{\mathcal{C}}E\left\{K_{h1}(x)K_{h1}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}(Y_j-m(X_1))\right]^4\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&\leq 8\int_{\mathcal{C}}E\left\{K_{h1}(x)K_{h1}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}(m(X_j)-m(X_1))\right]^4\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&\quad +8\int_{\mathcal{C}}E\left\{K_{h1}(x)K_{h1}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hj1}(\varepsilon_j\sqrt{v_0(X_j)})\right]^4\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&:= O\Big(\frac{1}{n^3h^{3d}}\Big)+O(h^8).
\end{aligned}
$$

With similar technique, we also have

$$
\begin{aligned}
EnS_{n62} &= n\int_{\mathcal{C}}E\left\{\frac{1}{n^2}\sum_{i\neq k}^{n}K_{hi}(x)K_{hj}(y)\left[\frac{1}{n}\sum_{j=1}^{n}K_{hij}(Y_j-m(X_i))\right]^2\right.\\
&\qquad\left.\cdot\left[\frac{1}{n}\sum_{k=1}^{n}K_{hki}(Y_k-m(X_i))\right]^2\right\}\dot{\mu}(x,\theta_0)\dot{\mu}(y,\theta_0)d\varphi(x)d\varphi(y)\\
&= O(nh^8)+O\left(\frac{1}{nh^{2d-4}}\right)+O\left(\frac{1}{n^3h^{4d}}\right)=o_p(1).
\end{aligned}
$$

The last inequality is obtained from assumption (h2), which further induce the dimension restriction.

Next, we focus on $n\tilde{S}^2_{n8}$ to $n\tilde{S}^2_{n12}$, and $n\tilde{S}^2_{n14}$ to $n\tilde{S}^2_{n18}$ first. Since we have shown that $n\tilde{S}^2_{n2}$ to $n\tilde{S}^2_{n6}$ are $o_p(1)$, it would be easy to show that all those terms are bounded above by $n\tilde{S}^2_{n2}$ to $n\tilde{S}^2_{n6}$, respectively, and then it suffices to show that $n\tilde{S}^2_{n7}$ and $n\tilde{S}^2_{n13}$ are $o_p(1)$. $n\tilde{S}^2_{n1}$ will be of order $O_p(1)$.

$$\begin{aligned}
\sqrt{n}\tilde{S}_{n8} &= \int_{\mathcal{C}} \sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[\xi(X_i)f(X_i)\Delta_{if}]\right\}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}^2\right\}d\hat{\varphi}_w(x) \\
&= \int_{\mathcal{C}} \sqrt{n}\left\{\frac{1}{n^2}\sum_{i=1}^n K_{hi}^2(x)\xi(X_i)f(X_i)\Delta_{if}^3\dot{v}(X_i,\theta_0)\right. \\
&\quad \left. + \frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}^n K_{hi}(x)K_{hj}(x)\xi(X_i)f(X_i)\dot{v}(X_j,\theta_0)\Delta_{if}^2\Delta_{jf}\right\}d\hat{\varphi}_w(x) := S_{n81}+S_{n82}.
\end{aligned}$$

Note that $ES_{n81} = ES_{n82} = 0$, so it suffices to show $E(S^2_{n81}) = o(1)$ and $E(S^2_{n82}) = o(1)$. By Fubini's Theorem, and the fact that $E(\xi(X_1)|X_1) = 0$, $E(S^2_{n81})$ equals

$$\begin{aligned}
&\frac{1}{n^3}\int_{\mathcal{C}} E\Big(\sum_{i=1}^n K_{hi}^2(x)\xi(X_i)f(X_i)\dot{v}(X_i,\theta)\Big(\frac{1}{n}\sum_{j=1}^n K_{hij} - f(X_i)\Big)^3\Big)^2 d\hat{\varphi}_w(x) \\
&= \frac{1}{n^3}\int_{\mathcal{C}} nE\Big(K_{h1}^4(x)\xi^2(X_1)f^2(X_1)\dot{v}^2(X_1,\theta)\Big(\frac{1}{n}\sum_{j=1}^n K_{h1j} - f(X_1)\Big)^6\Big)d\hat{\varphi}_w(x) \\
&= \frac{1}{n^2}\int_{\mathcal{C}} EK_{h1}^4(x)\xi^2(X_1)f^2(X_1)\dot{v}^2(X_1,\theta)\Big(\frac{1}{n}\sum_{j=1}^n K_{h1j} - E(K_{h1j}|X_1) \\
&\qquad\qquad + E(K_{h1j}|X_1) - f(X_1)\Big)^6 d\hat{\varphi}_w(x) \\
&\le \frac{32}{n^2}\int_{\mathcal{C}} EK_{h1}^4(x)\xi^2(X_1)f^2(X_1)\dot{v}^2(X_1,\theta)\Big(\frac{1}{n}\sum_{j=1}^n K_{h1j} - E(K_{h1j}|X_1)\Big)^6 d\hat{\varphi}_w(x) \\
&\qquad\qquad + \frac{32}{n^2}\int_{\mathcal{C}} EK_{h1}^4(x)\xi^2(X_1)f^2(X_1)\dot{v}^2(X_1,\theta)(E(K_{h1j}|X_1) - f(X_1))^6 d\hat{\varphi}_w(x) \\
&= \frac{32}{n^8}\int_{\mathcal{C}} EK_{h1}^4(x)\xi^2(X_1)f^2(X_1)\dot{v}^2(X_1,\theta)\Big(\sum_{j=1}^n (K_{h1j} - E(K_{h1j}|X_1))^6 \\
&\quad + \sum_{j\neq k\neq l}^n (K_{h1j} - E(K_{h1j}|X_1))^2(K_{h1k} - E(K_{h1k}|X_1))^2(K_{h1l} - E(K_{h1l}|X_1))^2 \\
&\quad + \sum_{j\neq k}^n (K_{h1j} - E(K_{h1j}|X_1))^4(K_{h1k} - E(K_{h1k}|X_1))^2\Big)d\hat{\varphi}_w(x) + O(h^{12}/n^2) \\
&= O\Big(\frac{1}{n^7h^{7d}}\Big) + O\Big(\frac{1}{n^6h^{6d}}\Big) + O\Big(\frac{1}{n^5h^{5d}}\Big) + O\Big(\frac{h^{12}}{n^2}\Big).
\end{aligned}$$

For $S_{n82}$, we have

$$E(S_{n82}^2) = \frac{1}{n^3}\int_{\mathcal{C}} E\Big(\sum_{i=1}^n\sum_{j\neq i}^n K_{hi}(x)K_{hj}(x)\xi(X_i)f(X_i)\dot{v}(X_i,\theta)\Delta_{if}\Delta_{jf}^2\Big)^2 d\hat{\varphi}_w(x)$$
$$= \frac{1}{n^3}\int_{\mathcal{C}} E\Big(\sum_{i=1}^n\sum_{j\neq i}^n K_{hi}^2(x)K_{hj}^2(x)\xi^2(X_i)f^2(X_i)\dot{v}^2(X_i,\theta)\Delta_{if}^2\Delta_{jf}^4$$
$$+\sum_{i=1}^n\sum_{j\neq i}^n\sum_{k\neq j\neq i}^n K_{hi}^2(x)K_{hj}(x)K_{hk}(x)\xi^2(X_i)f^2(X_i)\dot{v}(X_j,\theta)\dot{v}(X_k,\theta)\Delta_{if}^2\Delta_{jf}^2\Delta_{kf}^2\Big)d\hat{\varphi}_w(x)$$
$$= O\Big(\frac{1}{n^4h^{4d}}\Big)+O\Big(\frac{1}{n^3h^{3d}}\Big)+O\Big(\frac{h^{12}}{n}\Big)+O(h^{12}).$$

Thus we have,

$$\left(\sqrt{n}\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\xi(X_i)f(X_i)\Delta_{if}\right\}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\theta)\Delta_{if}f(X_i)\right\}d\hat{\varphi}_w(x)\right)^2 = o_p(1).$$

Next we will show $n\tilde{S}_{n13}^2$ converges to zero, and $n\tilde{S}_{n7}^2 = o_p(1)$ will follow similarly. Note that

$$n\tilde{S}_{n13}^2 = n\left(\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[\xi(X_i)f^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\varphi(x)\right)^2$$
$$= n\int_{\mathcal{C}}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[\xi(X_i)f^2(X_i)]\right\}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(y)[\xi(X_i)f^2(X_i)]\right\}$$
$$\cdot\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(y)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\varphi(x)d\varphi(y)$$
$$= n\int_{\mathcal{C}}\left\{\frac{1}{n^2}\sum_{i=1}^n K_{hi}(x)K_{hi}(y)\xi^2(X_i)f^4(X_i)]+\frac{1}{n^2}\sum_{i\neq j}^n K_{hi}(x)K_{hj}(y)\xi(X_i)\xi(X_j)f^2(X_i)f^2(X_j)\right\}$$
$$\cdot\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(y)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\varphi(x)d\varphi(y).$$

So, we have

$$En\tilde{S}_{n13}^2 = \int_{\mathcal{C}} EK_{h1}(x)K_{h1}(y)\xi^2(X_1)f^4(X_1)\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}$$
$$\cdot\left\{\frac{1}{n}\sum_{i=1}^n K_{hi}(y)\dot{v}(X_i,\theta)f(X_i)\Delta_{if}\right\}d\varphi(x)d\varphi(y)$$

which has the order of $o(1)$.

In the following we show that $\sqrt{n}S_{n1}$ is asymptotically normal. For the sake of brevity, we only prove the result for $d = 1$. The multidimensional version can be easily derived by using Wold technique. Rewrite $S_{n1}$,

$$\begin{aligned} S_{n1} &= \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \{\dot{\mu}_n(x,\theta) - \dot{\mu}_n(x,\theta_0) + \dot{\mu}_n(x,\theta_0) - \dot{\mu}_h(x,\theta_0) + \dot{\mu}_h(x,\theta_0)\} \\ &\qquad\qquad\qquad\qquad \{\hat{f}_w^{-6}(x) - f^{-6}(x) + f^{-6}(x)\} dG(x) \\ &= \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \dot{\mu}_h(x,\theta_0) f^{-6}(x) dG(x) \\ &\quad + \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \{\dot{\mu}_n(x,\theta) - \dot{\mu}_n(x,\theta_0)\} f^{-6}(x) dG(x) \\ &\quad + \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \{\dot{\mu}_n(x,\theta_0) - \dot{\mu}_h(x,\theta_0)\} f^{-6}(x) dG(x) \\ &\quad + \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \dot{\mu}_h(x,\theta_0) \{\hat{f}_w^{-6}(x) - f^{-6}(x)\} dG(x) \\ &\quad + \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \{\dot{\mu}_n(x,\theta) - \dot{\mu}_n(x,\theta_0)\} \{\hat{f}_w^{-6}(x) - f^{-6}(x)\} dG(x) \\ &\quad + \int_{\mathcal{C}} \left\{ \frac{1}{n} \sum_{i=1}^{n} K_{hi}(x)\xi(X_i) f^2(X_i) \right\} \{\dot{\mu}_n(x,\theta_0) - \dot{\mu}_h(x,\theta_0)\} \{\hat{f}_w^{-6}(x) - f^{-6}(x)\} dG(x) \\ &:= S_{n1,1} + S_{n1,2} + S_{n1,3} + S_{n1,4} + S_{n1,5} + S_{n1,6}. \end{aligned}$$

To proceed, we need the following lemmas:

**Lemma 2.1.** *Under $H_0$, suppose conditions (e1), (e2), (f1), (k), (v1) to (v5) hold:*

*(a). If additionally, (g) holds, $\sqrt{n} S_{n1,1} \to N(0, \Sigma)$.*

*(b). If (f2) and (w) hold, the $\sqrt{n}|S_{n1,4}| = o_p(1)$.*

**Lemma 2.2.** *Under $H_0$, suppose all the conditions in Lemma 2.1, Furthermore, assume (f2) and (w) hold, then $\sqrt{n} S_{n1,k} = o_p(1)$ holds for $k = 2, 3, 5, 6$.*

We also need the following lemma from Bosq (2012).

**Lemma 2.3.** *Let $\hat{f}_w$ be the kernel estimate associate with a kernel $K$ which satisfies a Lipschitz condition. If (f2) holds and $w_n = a_n (\log n/n)^{1/(d+4)}$, where $a_n \to a_0 > 0$, then*

$$(\log_k n)^{-1} (n/\log n)^{2/(d+4)} \sup_{x \in \mathcal{C}} |\hat{f}_w(x) - f(x)| \to 0, \quad a.s., \quad \forall k > 0, k \in N$$

To prove part (a) of Lemma 2.1, let $s_{n11i} := \int_{\mathcal{C}} K_{hi}(x)\xi(X_i) f^2(X_i) \dot{\mu}_h(x,\theta_0) d\varphi(x)$, we have

$$\sqrt{n} S_{n1,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_{n11i}$$

Note that $\{s_{n11i}, 1 \le i \le n\}$ are i.i.d. centered random variables for each n. By Central Limit Theorem with Lindeberg Feller condition, it suffices to show that as $n \to \infty$,

$$Es^2_{n111} \to \Sigma \tag{2.4}$$
$$E\{s^2_{n111} I(|s_{n111}| \ge n^{1/2}\eta)\} \to 0 \quad \forall \eta \ge 0 \tag{2.5}$$

By the definition of $\dot{\mu}_h(x,\theta)$, Fubini's theorem, change of variables by $\frac{u-x}{h} = u_1$, and $\frac{u-y}{h} = u_2$ and the continuity of functions $K, v, f, \dot{v}$, together with bounded convergence theorem, we have $\lim_{n\to\infty} \dot{\mu}_h(x,\theta) = \dot{v}_0(x) f^3(x)$, thus

$$\begin{aligned} Es^2_{n111} &= \int\int\int K_h(u-x)K_h(u-y)E(\xi^2|X=u)f^4(u)f(u)du\dot{\mu}_h(x,\theta_0)\dot{\mu}_h(y,\theta_0) \\ &\qquad \cdot f^{-6}(x)f^{-6}(y)g(x)g(y)dxdy \\ Es^2_{n111} &= \int\int\int K(u_1)K(u_2)\dot{\mu}_h(u+u_1h,\theta_0)\dot{\mu}_h(u+u_2h,\theta_0)f^{-6}(u+u_1h)f^{-6}(u+u_2h) \\ &\qquad \cdot g(u+u_1h)g(u+u_2h)du_1du_2f^5(u)E(\xi^2|X=u)du \\ &\to \int \dot{v}^2(u,\theta_0)f^6(u)f^{-6}(u)f^{-6}(u)f^5(u)g^2(u)E(\xi^2|X=u)du \\ &= \int \frac{\dot{v}^2(u,\theta_0)E(\xi^2|X=u)g^2(u)}{f(u)}du = \Sigma \end{aligned}$$

To prove the next claim, first note that by Holder's inequality,

$$\begin{aligned} E\{s^2_{n11}I(|s_{n11}| \ge n^{1/2}\eta)\} &= E(s^{2+\delta}_{n11})\frac{1}{s^{\delta}_{n11}}I(|s_{n11}| \ge n^{1/2}\eta) \le E\Big(s^{2+\delta}_{n11}\frac{1}{n^{\delta/2}\eta^{\delta}_d}\Big) \\ &= n^{-\frac{\delta}{2}}\eta_d^{-\delta}E\Bigg[\bigg(\int (K_{h1}(x)f^2(X_1)\dot{\mu}_h(x))^{(2+\delta)/2}d\varphi(x)\bigg)^2|\xi|^{2+\delta}\Bigg] = O(nh^d)^{-\delta/2} = o(1). \end{aligned}$$

To prove part (b) of the Lemma 2.1, we need to use Lemma 2.3, and the Cauchy-Schwartz inequality, the boundedness of $\dot{\mu}_h(x)$, we have,

$$\begin{aligned} nS^2_{n1,1} &\le Mn\int (\frac{1}{n}\sum_{i=1}^n K_{hi}(x)f^2(X_i)\xi(X_i)\dot{\mu}_h(x))^2 d\varphi(x)\sup_{x\in\mathcal{C}}|f^6(x)/\hat{f}^6_w(x) - 1|^2 \\ &= O_p((\log_k n)^2(\log n)^{4/(4+d)}n^{ad-4/(4+d)}) = o_p(1). \end{aligned}$$

And this concludes the proof of Lemma 2.1. Next, to show Lemma 2.2 we have, by Cauchy-Schwartz inequality,

$$||\sqrt{n}S_{n1,3}||^2 \le \Bigg(\sqrt{n}\int\Big(\frac{1}{n}\sum_{i=1}^n K_{hi}(x)f^2(X_i)\xi(X_i)\Big)^2 d\varphi(x)\Bigg)\Bigg(\sqrt{n}\int ||\dot{\mu}_n(x,\theta_0) - \dot{\mu}_h(x,\theta_0)||^2 d\varphi(x)\Bigg)$$

and it is again, easy to show that

$$E\sqrt{n}\int\Big(\frac{1}{n}\sum_{i=1}^n K_{hi}(x)f^2(X_i)\xi(X_i)\Big)^2 d\varphi(x) = O\Big(\frac{1}{n^{1/2}h^d}\Big) = o(1)$$

As for the second term, since $\dot{\mu}_h(x,\theta_0)$ is the mean of $\dot{\mu}_n(x,\theta_0)$, then the expectation of the second term would be bounded above by the second order of $\dot{\mu}_n(x,\theta_0)$, that is

$$\sqrt{n}\int E||K_{h1}(x)\dot{v}_0(x)f^2(X_1)||^2 d\varphi(x) = O\Big(\frac{1}{n^{1/2}h^d}\Big) = o(1)$$

Similarly we can ge the proof of $\sqrt{n}S_{n1,6}$.

$$||\sqrt{n}S_{n1,2}||^2 \le n\int \Big(\frac{1}{n}\sum_{i=1}^n K_{hi}(x)f^2(X_i)\xi(X_i)\Big)^2 d\varphi(x)\int ||\dot{\mu}_n(x,\hat{\theta}_n) - \dot{\mu}_n(x,\theta_0)||^2 d\varphi(x)$$

by assumption (v5), we know the second part is bounded above by

$$\max_{1\le i\le n}||\dot{v}(X_i,\theta_0) - \dot{v}(\theta_n, X_i)||^2\int (\hat{f}_h(x)f^2(x))^2 d\varphi(x) = o_p(h^d)O_p(1)$$

Together, we can show that $\sqrt{n}S_{n1,2} = o_p(1)$, and $\sqrt{n}S_{n1,5} = o_p(1)$ follows similarly. The proof of $\sqrt{n}S_{n1,6}$ follows $\sqrt{n}S_{n1,6}$ and $\sqrt{n}S_{n1,2}$, which complete the proof of both lemmas, and therefore the left hand side of equation (2.3).

Similarly, for the RHS of equation (2.3), plus and minus the true density at sample point $X_i$, we have the following,

$$\begin{aligned}
&\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))\hat{f}_h^2(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\hat{f}_h^2(X_i)\Bigg]d\hat{\varphi}_w(x)\\
=&\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))f^2(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)f^2(X_i)\Bigg]d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))f^2(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\Delta_{if}f(X_i)\Bigg]d\hat{\varphi}_w(x)\\
&+\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))f^2(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\Delta_{if}^2\Bigg]d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))\Delta_{if}f(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)f^2(X_i)\Bigg]d\hat{\varphi}_w(x)\\
&+4\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))\Delta_{if}f(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\Delta_{if}f(X_i)\Bigg]d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))\Delta_{if}f(X_i)\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\Delta_{if}^2\Bigg]d\hat{\varphi}_w(x)\\
&+\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))\Delta_{if}^2\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)f^2(X_i)\Bigg]d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot{v}_0(X_i) - \dot{v}(X_i,\hat{\theta}_n))\Delta_{if}^2\Bigg]\Bigg[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\Delta_{if}f(X_i)\Bigg]d\hat{\varphi}_w(x)
\end{aligned}$$

$$+\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)(\dot{v}_0(X_i)-\dot{v}(X_i,\hat{\theta}_n))\Delta_{if}^2\right]\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)\Delta_{if}^2\right]d\hat{\varphi}_w(x).$$

Note that with the previous derivation, we may conclude that except the first one, the remaining eight terms are $o_p(1)$. So the first term may have the following expansion,

$$\begin{aligned}
&\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)(\dot{v}_0(X_i)-\dot{v}(X_i,\hat{\theta}_n))f(X_i)^2\right]\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\dot{v}(X_i,\hat{\theta}_n)f(X_i)^2\right]d\hat{\varphi}_w(x)\\
&=\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)(\dot{v}_0(X_i)-\dot{v}(X_i,\hat{\theta}_n)-u_n^T\dot{v}_0(X_i)+u_n^T\dot{v}_0(X_i))f^2(X_i)\right]\dot{\mu}_n(x,\hat{\theta}_n)d\hat{\varphi}_w(x)\\
&=\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\Big(\frac{\dot{v}_0(X_i)-\dot{v}(X_i,\hat{\theta}_n)-u_n^T\dot{v}_0(X_i)}{||\hat{\theta}_n-\theta_0||}\Big)f^2(X_i)\right]\dot{\mu}_n(x,\hat{\theta}_n)d\hat{\varphi}_w(x)||u_n||\\
&\qquad\qquad+\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)(u_n^T\dot{v}_0(X_i))f^2(X_i)\right]\dot{\mu}_n(x,\hat{\theta}_n)d\hat{\varphi}_w(x)\\
&=\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)\frac{d_{ni}}{||u_n||}f^2(X_i)\right]\dot{\mu}_n(x,\hat{\theta}_n)d\hat{\varphi}_w(x)||u_n||+\int_{\mathcal{C}}\dot{\mu}_n(x,\theta_0)\dot{\mu}_n^T(x,\hat{\theta}_n)d\hat{\varphi}_w(x)u_n\\
&:=V_n||u_n||+L_nu_n.
\end{aligned}$$

By assumptions (v4) and (v5),

$$||V_n||\le\max_{1\le i\le n}\frac{d_{ni}}{||u_n||}\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)f^2(X_i)\right]||\dot{\mu}_n(x,\hat{\theta}_n)-\dot{\mu}_h(x,\theta_0)+\dot{\mu}_h(x,\theta_0)||d\hat{\varphi}_w(x)$$

$$\begin{aligned}
&\le\max_{1\le i\le n}\frac{d_{ni}}{||u_n||}||\dot{v}(X_i,\hat{\theta}_n)-\dot{v}_0(X_i)||\int_{\mathcal{C}}\hat{f}_h(x)f^2(x)d\hat{\varphi}_w(x)+\int_{\mathcal{C}}\hat{f}_h(x)f^2(x)||\dot{\mu}_h(x,\theta_0)||d\hat{\varphi}_w(x)\\
&=o_p(1)(o_p(1)+O_p(1))=o_p(1),
\end{aligned}$$

this concludes the proof of $V_n=o_p(1)$.

$$\begin{aligned}
L_n&=\int_{\mathcal{C}}\dot{\mu}_n(x,\theta_0)(\dot{\mu}_n(x,\hat{\theta}_n)-\dot{\mu}_h(x,\theta_0)+\dot{\mu}_h(x,\theta_0))^Td\hat{\varphi}_w(x)\\
&=\int_{\mathcal{C}}\dot{\mu}_n(x,\theta_0)(\dot{\mu}_n(x,\hat{\theta}_n)-\dot{\mu}_h(x,\theta_0))^Td\hat{\varphi}_w(x)+\int_{\mathcal{C}}\dot{\mu}_n(x,\theta_0)\dot{\mu}_h(x,\theta_0)^Td\hat{\varphi}_w(x)\\
&:=L_{n1}+L_{n2}.
\end{aligned}$$

And it is easy to see that $||L_{n1}||=o_p(1)$ by assumption (v5). Similarly by adding and subtracting $\dot{\mu}_h(x,\theta_0)$ from $\dot{\mu}_n(x,\theta_0)$ in $L_{n2}$, we have,

$$\begin{aligned}
||L_{n2}-\int_{\mathcal{C}}\dot{\mu}_h^T(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\hat{\varphi}_w(x)||&\le\int_{\mathcal{C}}||\dot{\mu}_n(x,\theta_0)-\dot{\mu}_h(x,\theta_0)||^2d\hat{\varphi}_w(x)\\
&+2\int_{\mathcal{C}}||\dot{\mu}_n(x,\theta_0)-\dot{\mu}_h(x,\theta_0)||\cdot||\dot{\mu}_h(x,\theta_0)||d\hat{\varphi}_w(x)=o_p(1).
\end{aligned}$$

Note that

$$\int_{\mathcal{C}} \dot{\mu}_h^T(x,\theta_0)\dot{\mu}_h(x,\theta_0)d\hat{\varphi}_w(x) = \int_{\mathcal{C}} \dot{v}(\theta_0)f^3(x)\dot{v}^T(\theta_0)f^3(x)d\hat{\varphi}_w(x) + o_p(1).$$

and $\int_{\mathcal{C}} \dot{v}(\theta_0)f^3(x)\dot{v}^T(\theta_0)f^3(x)d\hat{\varphi}_w(x)$ equals

$$\int_{\mathcal{C}} \dot{v}(\theta_0)f^3(x)\dot{v}^T(\theta_0)f^3(x)\Big\{\frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)}\Big\}dG(x) + \int_{\mathcal{C}} \dot{v}(\theta_0)\dot{v}^T(\theta_0)dG(x) = o_p(1) + \Sigma_0,$$

this concludes the proof. □

To prove Theorem 3.4, we need the Theorem 1 from Hall (1984) which is reproduced here for the sake of completeness.

**Lemma 2.4.** *Let $\tilde{X}_i$, $1 \le i \le n$, be i.i.d. random vectors, and define*

$$U_n := \sum_{1\le i<j\le n} H_n(\tilde{X}_i, \tilde{X}_j), \quad G_n(x,y) = EH_n(\tilde{X}_1, x)H_n(\tilde{X}_1, y),$$

*where $H_n$ is a sequence of measurable functions symmetric under permutation, with*

$$E(H_n(\tilde{X}_1, \tilde{X}_2)|\tilde{X}_1) = 0 \ \ a.s., \quad EH_n^2(\tilde{X}_1, \tilde{X}_2) < \infty$$

*for each $n \ge 1$. If*

$$\frac{[EG_n^2(\tilde{X}_1, \tilde{X}_2) + n^{-1}EH_n^4(\tilde{X}_1, \tilde{X}_2)]}{[EH_n^2(\tilde{X}_1, \tilde{X}_2)]^2} \to 0,$$

*then $U_n$ is asymptotically normal with mean 0 and variance $\lim_{n\to\infty} n^2 EH_n^2(\tilde{X}_1, \tilde{X}_2)/2$.*

The proof of Theorem 3.4 is facilitated by the following lemmas.

**Lemma 2.5.** *Under $H_0$, if (e1), (e2), (e4), (f1), (g), (h1), (h2), and (k) hold, then*

$$nh^{d/2}(\tilde{T}_n(\theta_0) - \tilde{C}_n) \to N(0, \Gamma)$$

*in distribution.*

*Proof.* First, note that by the similar derivation as in the proof of asymptotic normality of $\hat{\theta}_n$, we have

$$nh^{d/2}T_n(\theta_0) = nh^{d/2}\int_{\mathcal{C}} \Bigg[\frac{1}{n}\sum_{i=1}^{n} K_{hi}(x)[(Y_i - m(X_i) + m(X_i) - \hat{m}(X_i))^2 - v_0(X_i)]$$
$$\cdot(\hat{f}_h(X_i) - f(X_i) + f(X_i))^2\Bigg]^2 \left(\frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} + \frac{1}{f^6(x)}\right)dG(x)$$

From the previous results, under similar assumptions, together with the dimension restriction $d < 4$, it would be easy to see that the dominating term on the right hand side is $nh^{d/2}\int_{\mathcal{C}}[n^{-1}\sum_{i=1}^{n} K_{hi}(x)[(Y_i - m(X_i))^2 - v_0(X_i)]f^2(X_i)]^2 d\varphi(x)$, denoted as $\tilde{T}_n(\theta_0)$.

It can be seen that

$$nh^{d/2}\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^{n}K_{hi}(x)[(Y_i-m(X_i))^2-v_0(X_i)]f^2(X_i)\right]^2 d\varphi(x)$$
$$=nh^{d/2}\int_{\mathcal{C}}\frac{1}{n^2}\sum_{i=1}^{n}K_{hi}^2(x)\xi^2(X_i)f^4(X_i)d\varphi(x)$$
$$+\int_{\mathcal{C}}\frac{1}{n^2}\sum_{i\neq j}^{n}K_{hi}(x)K_{hj}(x)\xi(X_i)\xi(X_j)f^2(X_i)f^2(X_j)d\varphi(x).$$

We denote the right hand side as $nh^{d/2}(\tilde{C}_n+M_n)$. Also define

$$\Gamma_n := 2h^d\frac{n-1}{n}\int\int[EK_{h1}(x)K_{h1}(y)\xi^2(X_i)f^4(X_1)]^2 d\varphi(x)d\varphi(y)$$
$$H_n(\tilde{X}_i,\tilde{X}_j) := \frac{h^{d/2}}{n}\int K_{hi}(x)K_{hj}(x)\xi(X_i)\xi(X_j)f^2(X_i)f^2(X_j)d\varphi(x),$$

where $\tilde{X}_i=(X_i^T,\xi(X_i))^T$. So $nh^{d/2}M_n/2=\sum_{i<j}H_n(\tilde{X}_i,\tilde{X}_j)$, and we will have,

$$EH_n^2(\tilde{X}_1,\tilde{X}_2)=\frac{h^d}{n^2}\iint EK_{hi,j}(x)K_{hi,j}(y)\xi^2(X_i)\xi^2(X_j)f^4(X_i)f^4(X_j)d\varphi(x)d\varphi(y)$$
$$=\frac{h^d}{n^2}\iint[EK_{h1}(x)K_{h1}(y)\xi^2(X_1)f^4(X_1)]^2 d\varphi(x)d\varphi(y)$$
$$=\frac{h^d}{n^2}\iint\left(\int\frac{1}{h^{2d}}K\left(\frac{u-x}{h}\right)K\left(\frac{u-y}{h}\right)\xi^2(u)f^4(u)f(u)du\right)^2 d\varphi(x)d\varphi(y)$$
$$=\frac{1}{n^2h^d}\iint\left(K(u)K\left(\frac{x-y}{h}+u\right)\xi^2(x+uh)f^5(x+uh)du\right)^2 d\varphi(x)d\varphi(y)$$

Next, we will show

$$\frac{EG^2(\tilde{X}_1,\tilde{X}_2)}{[EH_n^2(\tilde{X}_1,\tilde{X}_2)]^2}=o(1),\quad \frac{EH^4(\tilde{X}_1,\tilde{X}_2)}{n[EH_n^2(\tilde{X}_1,\tilde{X}_2)]^2}=o(1),$$

which implies the desired result by Lemma 2.4. In fact, the above results can be obtained if we can show

$$EG^2(\tilde{X}_1,\tilde{X}_2)=O\Big(\frac{h^d}{n^4}\Big),\quad EH_n^2(\tilde{X}_1,\tilde{X}_2)=O\Big(\frac{1}{n^2}\Big),\quad EH^4(\tilde{X}_1,\tilde{X}_2)=O\Big(\frac{1}{n^4h^{2d}}\Big).$$

Note that $G_n(x,y)=EH_n(\tilde{X}_1,x)H_n(\tilde{X}_1,y)$, left $t^T=(t_1^T,t_2)$, and $s^T=(s_1^T,s_2)$, where $t_1$ and $s_1$

are $n \times 1$ vector, where $t_2$ and $s_2$ are scalar. So, we can get

$$
\begin{aligned}
&G_n(t,s) = EH_n(\tilde{X}_1,t)H_n(\tilde{X}_1,s)\\
&= h^d n^{-2} E\int K_h(\tilde{X}_1 - x)K_h(t_1 - x)\xi(X_1)t_2 f^2(X_1)f^2(t_1)K_h(\tilde{X}_1 - z)K_h(s_1 - z)\xi(X_1)s_2\\
&\qquad \cdot f^2(X_1)f^2(s_1)d\varphi(x)d\varphi(y)\\
&= h^d n^{-2}\int K_h(t_1 - x)t_2 s_2 f^2(t_1)K_h(s_1 - z)f^2(s_1)\\
&\qquad \cdot E(K_h(\tilde{X}_1 - x)K_h(\tilde{X}_1 - z)\xi^2(X_1)f^4(X_1))d\varphi(x)d\varphi(y)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&EK_{h1}(x)K_{h1}(z)\xi^2(X_1,\hat{\theta}_n)f^4(X_1) = \int K_h(u - x)K_h(u - z)\xi^2(u)f^5(u)du\\
&= \frac{1}{h^d}\int K(u)K\Big(\frac{x - z}{h} + u\Big)\xi^2(x + uh)f^5(x + uh)du := B_h(x - z)
\end{aligned}
$$

$$
EG_n^2(\tilde{X}_1,\tilde{X}_2) = \frac{h^{2d}}{n^4}\int B_h(x - z)B_h(x - w)B_h(z - v)B(v - w)d\varphi(x)d\varphi(z)d\varphi(w)d\varphi(v)
$$

which is the order of $n^{-4}h^d$, where the last equality hold by change of variables on $(x - z)/h = u$, $(x - w)/h = v$, $(z - v)/h = w$, and the continuity of functions $K$, $\xi$ over compact set, we have

$$
\begin{aligned}
&EH_n^4(\tilde{X}_1,\tilde{X}_2) = \frac{h^{2d}}{n^4}E\left(\int K_{h1,2}(x)\xi(X_1)\xi(X_2)f^2(X_1)f^2(X_2)d\varphi(x)\right)^4\\
&= \frac{h^{2d}}{n^4}\iiiint (EK_{h1}(x)K_{h1}(y)K_{h1}(z)K_{h1}(t)\xi(X_1)^4 f^8(X_1))^2 d\varphi(x)d\varphi(y)d\varphi(z)d\varphi(t),
\end{aligned}
$$

which is the order of $O(n^{-4}h^{-2d})$. We also have

$$
\begin{aligned}
&EH_n^2(\tilde{X}_1,\tilde{X}_2) = \frac{h^d}{n^2}\iint EK_{h1,2}(x)K_{h1}(y)K_{h2}(y)\xi^2(X_1)\xi^2(X_2)f^4(X_1)f^4(X_2)d\varphi(x)d\varphi(y)\\
&= \frac{1}{n^2h^{3d}}\iiiint K\left(\frac{x - u}{h}\right)K\left(\frac{y - u}{h}\right)K\left(\frac{x - v}{h}\right)K\left(\frac{y - v}{h}\right)\xi^2(u)\xi^2(v)\\
&\qquad \cdot f^5(u)f^5(v)dudvd\varphi(x)d\varphi(y),
\end{aligned}
$$

which is $O(n^{-2})$.

So far we have shown that the conditions for the CLT are satisfied, and next it suffices to show the form of variance. In fact, we have

$$
\begin{aligned}
&\frac{1}{2}n^2 EH_n^2(\tilde{X}_1,\tilde{X}_2) = \frac{n\Gamma_n}{4(n - 1)}\\
&= \frac{1}{2}h^d\int\int\left(\int K(u)\frac{1}{h^d}K\left(\frac{y - x}{h} + u\right)\tau^2(x - uh)f^5(x - uh)du\right)^2 d\varphi(x)d\varphi(y)
\end{aligned}
$$

$$\to \frac{1}{2}\int (\tau^2(x))^2 f^{10}(x) f^{-12}(x) g^2(x)dx \int \left(\int K(u)K(v+u)du\right)^2 dv$$
$$= \frac{1}{2}\int \frac{(\tau^2(x))^2 g^2(x)}{f^2(x)}dx \int \left(\int K(u)K(v+u)du\right)^2 dv = \frac{\Gamma}{4},$$

by the Bounded Convergence theorem, and the continuity of $\tau^2(x)$, $f(x)$ and $g(x)$. □

**Lemma 2.6.** *Under $H_0$, if (e1), (e2), (f1), (f2), (k), (v3)-(v5) and (w) hold, then*

$$nh^{d/2}|T_n(\hat\theta_n) - T_n(\theta_0)| = o_p(1).$$

*Proof.* Note that $T_n(\theta_0) - T_n(\hat\theta_n)$ can be written as $2Q_1 - Q_2$ with

$$Q_1 = \int \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)((Y_i - \hat m(X_i))^2 - v_0(X_i))\hat f_h^2(X_i)\right]$$
$$\cdot \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(v_0(X_i) - v(X_i,\hat\theta_n))\hat f_h^2(X_i)\right] d\hat\varphi_w(x)$$
$$Q_2 = \int \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(v_0(X_i) - v(X_i,\hat\theta_n))\hat f_h^2(X_i)\right]^2 d\hat\varphi_w(x).$$

It suffices to show that $nh^{d/2}Q_1 = o_p(1)$, and $nh^{d/2}Q_2 = o_p(1)$. Note that $Q_1$ can be written as

$$\int U_n(x)\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(v_0(X_i) - v(X_i,\hat\theta_n) - (\hat\theta_n - \theta_0)^T \dot v_0(X_i) + (\hat\theta_n - \theta_0)^T \dot v_0(X_i))\hat f_h^2(X_i)\right] d\hat\varphi_w(x)$$

which is the sum of the following two terms

$$Q_{11} = \int U_n(x)\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x) d_{ni}\right]\hat f_h^2(X_i) d\hat\varphi_w(x),$$
$$Q_{12} = u_n^T \int U_n(x)\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot v_0(X_i)\right]\hat f_h^2(X_i) d\hat\varphi_w(x).$$

By Theorem 3.2, and $nh^d T_n(\theta_0) = O_p(1)$ we have $nh^d||\hat\theta_n - \theta_0||^2 = O_p(1)$. Therefore, by Cauchy-Schwartz inequality, assumption (v4), and the fact that $\hat f_h(X_i)/\hat f_h(x) = O_p(1)$, $\int U_n(x)d\varphi(x) = O(1/nh^d)$, we have

$$nh^{d/2}|Q_{11}| \le \max_i \frac{|d_{ni}|}{||u_n||}(nh^d)^{1/2}||u_n|| \int \hat f_h^6(x) d\hat\varphi_w(x)\sqrt{n}\int U_n^2(x) d\hat\varphi_w(x) = o_p(1)$$

Next, observe that $Q_{12}$ can be written as $Q_{121} - Q_{122}$, where

$$Q_{121} = u_n^T \int U_n(x)\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\dot v(X_i,\hat\theta_n)\right]\hat f_h^2(X_i) d\hat\varphi_w(x),$$
$$Q_{122} = u_n^T \int U_n(x)\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)(\dot v(X_i,\hat\theta_n) - \dot v_0(X_i))\right]\hat f_h^2(X_i) d\hat\varphi_w(x).$$

Arguing as above, on the event $(nh^d)^{1/2}||\hat{\theta}_n - \theta_0|| \le k$, with Cauchy-Schwartz inequality,

$$\begin{aligned} n^2h^d|Q_{122}|^2 \le n^2h^d||u_n||^2 \int U_n^2(x)d\hat{\varphi}_w(x) \cdot \max_i ||\dot{v}(\hat{\theta}_n, X_i) - \dot{v}(\theta_0, X_i)||^2 d\hat{\varphi}_w(x) \\ \cdot \int \left(\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\hat{f}_h^2(X_i)\right)^2 d\hat{\varphi}_w(x) \\ = n^2h^d O_p\left(\frac{1}{n^2h^{2d}}\right) O_p\left(\frac{1}{nh^d}\right) O_p(h^d)O_P(1) = o_p(1). \end{aligned}$$

For $Q_{121}$, we can further rewrite it as the sum of $D_1$ and $D_2$, where

$$D_1 = u_n^T \int Z_n(x, \hat{\theta}_n)\dot{\mu}_n(x, \theta_0)d\hat{\varphi}_w(x),$$
$$D_2 = u_n^T \int Z_n(x, \hat{\theta}_n)(\dot{\mu}_n(x, \hat{\theta}_n) - \dot{\mu}_n(x, \theta_0))d\hat{\varphi}_w(x),$$

and it is obvious to see that

$$nh^{d/2}D_1 \le nh^{d/2}||u_n||^2 \int Z_n^2(x, \hat{\theta}_n)d\hat{\varphi}_w(x) \int \hat{f}_h^2(x)d\hat{\varphi}_w(x) = o_p(1).$$

While using Theorem 3.5, and assumption (v5)

$$nh^{d/2}D_2 \le nh^{d/2}||u_n||^2 \int Z_n^2(x, \hat{\theta}_n)d\hat{\varphi}_w(x) \int \left[\frac{1}{n}K_{hi}(x)(\dot{v}(X_i, \hat{\theta}_n) - \dot{v}(X_i, \theta_0))\right]^2 d\hat{\varphi}_w(x),$$

which is $o_p(1)$. The proof of $Q_2$ will be proceeded similar to $Q_1$. □

**Lemma 2.7.** *Under $H_0$, (e1), (e2), (f1), (f2), (v3)-(v5), (w), we have*

$$nh^{d/2}|T_n(\theta_0) - \tilde{T}_n(\theta_0)| = o_p(1)$$

*Proof.* Using Lemma 2.3, we have

$$\begin{aligned} nh^{d/2}|T_n(\theta_0) - \tilde{T}_n(\theta_0)| \le nh^{d/2} \int_{\mathcal{C}} U_n^2(x)d\varphi(x) \sup_{x\in\mathcal{C}} |f^6(x)/\hat{f}_w^6(x) - 1| \\ = nh^{d/2}O_p((nh^d)^{-1})O_p((\log_k n)(\log n/n)^{d/(d+4)}) = o_p(1), \end{aligned}$$

which implies the desired result. □

**Lemma 2.8.** *Under the same condition as Lemma 2.6, $nh^{d/2}(\hat{C}_n - \tilde{C}_n) = o_p(1)$.*

*Proof.* Since

$$\hat{C}_n = \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)\hat{\xi}^2(X_i)\hat{f}_h^4(X_i)d\hat{\varphi}_w(x), \quad \tilde{C}_n = \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)\xi^2(X_i)f^4(X_i)d\varphi(x).$$

As usual, for $\hat{C}_n$, by adding and subtracting the following quantities, $m(X_i)$, $v_0(X_i)$, $f(X_i)$, and $f^{-6}(x)$ in the corresponding places, i.e. $\hat{\xi}^2(X_i)$ for the first two, and $\hat{f}_h^4(X_i)$, $d\hat{\varphi}_w(x)$, respectively. Let $\Delta_{iv} := v_0(X_i) - v(X_i, \hat{\theta}_n)$, we have

$$\begin{aligned}
\hat{C}_n &= \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)((Y_i - \hat{m}(X_i) - m(X_i) + m(X_i))^2 - v(X_i, \hat{\theta}_n) + v_0(X_i) - v_0(X_i)) \\
&\qquad (\hat{f}_h^4(X_i) - f^4(X_i) + f^4(X_i)) \left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} + \frac{1}{f^6(x)} \right) dG(x) \\
&= \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)\xi^2(X_i) f^4(X_i) d\varphi(x) + \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)\xi^2(X_i)\Delta_{if^4} d\varphi(x) \\
&+ \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)\xi^2(X_i)\Delta_{if^4}\left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} \right) dG(x) \\
&+ \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)\xi^2(X_i) f^4(X_i)\left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} \right) dG(x) \\
&+ \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)(\Delta_{im}^2)\Delta_{if^4}\left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} \right) dG(x) \\
&+ \frac{2}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)((Y_i - m(X_i))\Delta_{im} - \Delta_{iv})\Delta_{if^4}\left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} \right) dG(x) \\
&+ \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)(\Delta_{im}^2) f^4(X_i)\left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} \right) dG(x) \\
&+ \frac{2}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)((Y_i - m(X_i))\Delta_{im} - \Delta_{iv}) f^4(X_i)\left( \frac{1}{\hat{f}_w^6(x)} - \frac{1}{f^6(x)} \right) dG(x) \\
&+ \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)(\Delta_{im}^2)\Delta_{if^4} f^{-6}(x) dG(x) \\
&+ \frac{2}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)((Y_i - m(X_i))\Delta_{im} - \Delta_{iv})\Delta_{if^4} f^{-6}(x) dG(x) \\
&+ \frac{1}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)(\Delta_{im}^2) f^4(X_i) f^{-6}(x) dG(x) \\
&+ \frac{2}{n^2}\sum_{i=1}^n \int_{\mathcal{C}} K_{hi}^2(x)((Y_i - m(X_i))\Delta_{im} - \Delta_{iv}) f^4(X_i)\left( \frac{1}{f^6(x)} \right) dG(x) \\
&:= C_{n1} + \cdots + C_{n12}.
\end{aligned}$$

Note that $C_{n1}$ is simply $\tilde{C}_n$. For $i = 2 \ldots 12$, we will show that $nh^{d/2}C_{ni} = o_p(1)$. By Cauchy-Schwartz inequality, it is sufficient to show that $nh^{d/2}C_{nk} \to 0$ in probability for $k = 2, 4, 12$. For

$C_{n2}$, we have

$$Enh^{d/2}C_{n2} = h^{d/2}\int_{\mathcal{C}} EK_{h1}^2(x)\xi^2(X_1)(\hat{f}_h^4(X_1) - f^4(X_1))d\varphi(x)$$

$$= h^{d/2}\int_{\mathcal{C}} EK_{h1}^2(x)\xi^2(X_1)\left[\left(\frac{1}{n}\sum_{j=1}^{n} K_{hj1} - f(X_1) + f(X_1)\right) - f^4(X_1)\right]d\varphi(x) = o(h^{4+d/2}).$$

By Lemma 2.3, expanding the numerator in the fraction part, also note that other functions are all continuous on the compact set $\mathcal{C}$,

$$Enh^{d/2}|C_{n4}| \le h^{d/2}\int_{\mathcal{C}} EK_{h1}^2(x)\xi^2(X_1)f^4(x)\sup_{x\in\mathcal{C}}\left|\frac{f^6(x) - \hat{f}_w^6(x)}{\hat{f}_w^6(x)f^6(x)}\right|dG(x) = o(1).$$

Finally, for $C_{n12}$, it would be easy to see that

$$Enh^{d/2}C_{n12} = 2h^{d/2}\int_{\mathcal{C}} EK_{h1}^2(x)((Y_1 - m(X_1))\Delta_{1m} - \Delta_{1v})f^4(X_1)d\varphi(x)$$

$$= 2h^{d/2}\int_{\mathcal{C}} EK_{h1}^2(x)((Y_1 - m(X_1))\Delta_{1m})f^4(X_1)d\varphi(x) - 2h^{d/2}\int_{\mathcal{C}} EK_{h1}^2(x)\Delta_{1v}f^4(X_1)d\varphi(x),$$

is of order $o(1)$, while using similar techniques to $\Delta_{1m}$ and $\Delta_{1v}$. To be specific, the first part on the RHS is similar to $T_{n2}^*$ with a simpler derivation, and the second part on the RHS converge to zero with assumption $v(5)$ by taking the term $\max\Delta_{1v}$ outside the integral.

For $i = 3, 5, 6, 7, 8, 9, 10, 11$, we will observe similar structure as the above two, thus using similar technique as the above two terms may grant the order $o_p(1)$. □

**Lemma 2.9.** *Under the same conditions as Lemma 2.6, we have* $\hat{\Gamma}_n - \Gamma = o_p(1)$.

*Proof.* Since we have already known that $\Gamma_n \to \Gamma$, so if suffices to show that $\hat{\Gamma}_n - \tilde{\Gamma}_n = o_p(1)$, $\tilde{\Gamma}_n - \Gamma_n = o_p(1)$, where,

$$\tilde{\Gamma}_n = \frac{h^d}{n^2}\sum_{i\neq j}\left(\int_{\mathcal{C}} K_{hi}(x)K_{hj}(x)\xi(X_i)\xi(X_j)f_h^2(X_i)f_h^2(X_j)f^{-6}(x)dG(x)\right)^2.$$

First we show that $\hat{\Gamma}_n - \tilde{\Gamma}_n = o_p(1)$. Note that

$$\hat{\Gamma}_n = \frac{h^d}{n^2}\sum_{i\neq j}\left(\int_{\mathcal{C}} K_{hi}(x)K_{hj}(x)\hat{\xi}(X_i)\hat{\xi}(X_j)\hat{f}_h^2(X_i)\hat{f}_h^2(X_j)\hat{f}_w^{-6}(x)dG(x)\right)^2$$

$$= \frac{h^d}{n^2}\sum_{i\neq j}\Bigg(\int_{\mathcal{C}} K_{hi}(x)K_{hj}(x)((Y_i - m(X_i) - \Delta_{im})^2 - v_0(X_i) + v_0(X_i) - v(X_i,\hat{\theta}_n))\cdot$$

$$((Y_j - m(X_j) - \Delta_{jm})^2 - v_0(X_j) + v_0(X_j) - v(X_j,\hat{\theta}_n))\cdot$$

$$(\Delta_{if^2} + f^2(X_i))(\Delta_{if^2} + f^2(X_j))(\hat{f}_w^{-6}(x) - f^{-6}(x) + f^{-6}(x))dG(x)\Bigg)^2.$$

After expanding the square terms in the above expression of $\hat{\Gamma}_n$, $\hat{\Gamma}_n$ can be written as the sum of $\tilde{\Gamma}_n$ and many other terms which include at least one term from $\Delta^2_{im}$, $\Delta_{vi}$ and $\Delta_{if^2}$. Hence, except for $\tilde{\Gamma}_n$, all other terms will be $o_p(1)$.

Finally, note that by Fubini Theorem,

$$E\tilde{\Gamma}_n = \frac{h^d}{n^2}\sum_{i\neq j} E\left(\int_{\mathcal{C}} K_{hi}(x)K_{hj}(x)\xi(X_i)\xi(X_j)f_h^2(X_i)f_h^2(X_j)f^{-6}(x)dG(x)\right)^2$$
$$= \frac{(n-1)h^d}{n}\int\int (E(K_{h1}(x)K_{h1}(y)\xi^2(X_1)f^4(X_1)))^2 d\varphi(x)d\varphi(y) = \Gamma_n$$

Therefore, we have

$$E(\tilde{\Gamma}_n - \Gamma_n)^2 \leq \sum_{i\neq j} EH^4(\tilde{X}_i,\tilde{X}_j) + c\sum_{i\neq j\neq k} EH_n^2(\tilde{X}_i,\tilde{X}_j)H_n^2(\tilde{X}_j,\tilde{X}_k)$$
$$\leq (n^2+cn^3)EH^4(\tilde{X}_1,\tilde{X}_2) = O(1/nh^d).$$

This concludes the proof of Lemma 2.9. □

*The proof of Theorem 3.5.* Note that under the alternative hypothesis, we have $Y = m(X) + \sqrt{v_a(X)}\varepsilon$. So by adding and subtracting $v_a(X_i)$ in $T_n(\hat{\theta}_n)$, it can be written as $T_{nc1}+T_{nc2}+T_{nc3}$, where

$$T_{nc1} = \int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\{(Y_i-\hat{m}(X_i))^2 - v_a(X_i)\}\hat{f}_h^2(X_i)\right]^2 d\hat{\varphi}_w(x),$$
$$T_{nc2} = \int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\{v_a(X_i) - v(X_i,\hat{\theta}_n)\}\hat{f}_h^2(X_i)\right]^2 d\hat{\varphi}_w(x),$$

and $T_{nc3}$ is bounded above by $2\sqrt{T_{nc1}T_{nc2}}$ by Cauchy-Schwartz inequality.

By adding and subtracting $m(X_i)$, $f^2(X_i)$, and $f^{-6}(x)$ in the corresponding parts in $T_{nc1}$, and the dominating term would be

$$nh^{d/2}\tilde{T}_{nc1} = nh^{d/2}\int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\{(Y_i-m(X_i))^2 - v_a(X_i)\}f^2(X_i)\right]^2 d\varphi(x).$$

Denote $\xi_a(X_i) = (Y_i - m(X_i))^2 - v_a(X_i)$, and

$$T_{nc1} = \int_{\mathcal{C}}\left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)\{(Y_i-m(X_i))^2 - v_a(X_i)\}f^2(X_i)\right]^2 d\varphi(x),$$
$$\tilde{C}_{an} = \int_{\mathcal{C}}\left[\frac{1}{n^2}\sum_{i=1}^n K_{hi}^2(x)\xi_a^2(X_i)f^4(X_i)\right] d\varphi(x).$$

Then we can show that $nh^{d/2}(T_{nc1} - \tilde{C}_{an}) = nh^{d/2}(\tilde{T}_{nc1} - \tilde{C}_{an}) + o_p(1)$. Similar to the proof of Theorem 3.4, we further have $nh^{d/2}(T_{nc1} - \tilde{C}_{an}) \to N(0, \Gamma_a)$ with

$$\Gamma_a = 2\int (\xi_a^2(x))^2 f^{-2}(x) g^2(x) dx \int \left(\int K(u)K(u+v)du\right)^2 dv.$$

It is also easy to show that $\hat{\Gamma}_n \to \Gamma_a$. The details are omitted for the sake of brevity.

For $T_{nc2} \to \infty$, as $n \to \infty$, we can show that $T_{nc2} \to \int_{\mathcal{C}} (v_a(x) - v(x, \theta_a))^2 dG(x)$ in probability. As for $\hat{C}_n$ and $\tilde{C}_{an}$, we can easily show that they are both the order of $O_p(1/(nh^d))$. Therefore, we also have $|T_{nc3}| \le 2\sqrt{(T_{nc1} - \tilde{C}_{an} + \tilde{C}_{an})T_{nc2}} = o_p(1)$.

Finally, for $n$ large enough, we have

$$nh^{d/2}\hat{\Gamma}_n^{-1/2}|T_n(\hat{\theta}_n) - \hat{C}_n| = nh^{d/2}\hat{\Gamma}_n^{-1/2}|T_{nc1} - \tilde{C}_{an} + T_{nc2} + T_{nc3} + \tilde{C}_{an} - \hat{C}_n|,$$

which is bounded below by $nh^{d/2}\hat{\Gamma}_n^{-1/2}(T_{nc2} + o_p(1))$. This, together with the result $\hat{\Gamma}_n \to \Gamma$, implies the desired result. □

Now let us prove Theorem 3.6. Denote $v_{loc}(x, \theta_0) := v_0(x) + c_n\delta(x)$. Under the alternative hypothesis, we have $(Y - m(X))^2 = v_{loc}(X, \theta_0)\varepsilon^2$. The key part of the proof is to show that $nh^{d/2}(\tilde{T}_n(\theta_0) - \tilde{C}_n) \to N(\int_{\mathcal{C}} \delta^2(x)dG(x), \Gamma)$ in distribution and $nh^{d/2}(\hat{C}_n - \tilde{C}_{nL}) = o_p(1)$, where

$$\tilde{C}_{nL} := \int_{\mathcal{C}} \frac{1}{n^2}\sum_{i=1}^n K_{hi}^2(x)\xi_{loc}^2(X_i) f^4(X_i) d\varphi(x). \tag{2.6}$$

*Proof.* The proof largely follows the similar arguments in the proofs of previous theorem, hence only the main steps are sketched here for the sake of brevity. First we have to show that $nh^{d/2}(T_n(\hat{\theta}_n) - \tilde{T}_n(\theta_0)) = o_p(1)$, where

$$\tilde{T}_n(\theta_0) = \int_{\mathcal{C}} \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[(Y_i - m(X_i))^2 - v_0(X_i)] f^2(X_i)\right]^2 d\varphi(x).$$

Denote $(Y_i - m(X_i))^2 - v_{loc}(X_i, \theta_0) = \xi_{loc}(X_i)$, $\tilde{T}_n(\theta_0)$ can be further written as the sum of the following three terms

$$T_{nL1}(\theta_0) = \int_{\mathcal{C}} \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[(Y_i - m(X_i))^2 - v_{loc}(X_i, \theta_0)] f^2(X_i)\right]^2 d\varphi(x)$$

$$T_{nL2}(\theta_0) = \int_{\mathcal{C}} \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[v_{loc}(X_i, \theta_0) - v_0(X_i)] f^2(X_i)\right]^2 d\varphi(x)$$

$$T_{nL3}(\theta_0) = 2\int_{\mathcal{C}} \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[(Y_i - m(X_i))^2 - v_{loc}(X_i, \theta_0)] f^2(X_i)\right] \left[\frac{1}{n}\sum_{i=1}^n K_{hi}(x)[v_{loc}(X_i, \theta_0) - v_0(X_i)] f^2(X_i)\right] d\varphi(x).$$

For $T_{nL1}(\theta_0)$, it has the decomposition $nh^{d/2}T_{nL1}(\theta_0) = nh^{d/2}(\tilde{C}_{nL} + M_n)$, where

$$\tilde{C}_{nL} = \int_{\mathcal{C}} \frac{1}{n^2} \sum_{i=1}^{n} K_{hi}^2(x)\xi_{loc}^2(X_i) f^4(X_i) d\varphi(x)$$
$$M_n = \int_{\mathcal{C}} \frac{1}{n^2} \sum_{i \neq j} K_{hi}(x)K_{hj}(x)\xi_{loc}(X_i) f^2(X_i)\xi_{loc}(X_j) f^2(X_j) d\varphi(x).$$

We can further show that $nh^{d/2}(T_{nL1}(\theta_0) - \tilde{C}_{nL}) \xrightarrow{d} N(0, \Gamma)$, and $nh^{d/2}(\hat{C}_n - \tilde{C}_{nL}) = o_p(1), \hat{\Gamma}_n \to \Gamma$ under the local alternative hypothesis. Therefore, the desired result can be obtained after proving that $nh^{d/2}T_{nL2} \to \int_{\mathcal{C}} \delta^2(x)dG(x)$ in probability, and $nh^{d/2}T_{nL3} = o_p(1)$. As we mentioned before, the arguments are similar to those used in the proof of Theorem 3.4 and 3.5. $\square$

## References


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